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THEORETICAL STUDY OF THE PROPAGATION OF SOUND. APPLICATION TO ANTICIPATION OF THE SONIC BOOM PRODUCED BY SUPERSONIC FLIGHT

by Jean-Pierre Guiraud

National Aerospace Research and Development Administration
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THEORETICAL STUDY OF THE PROPAGATION OF SOUND.

APPLICATION TO ANTICIPATION OF THE SONIC

BOOM PRODUCED BY SUPERSONIC FLIGHT

By Jean-Pierre Guiraud

Notes of the Course on Higher Aerodynamics, at the
Henri-Poincaré Institute (Theoretical Mechanics)

NATIONAL AEROSPACE RESEARCH AND DEVELOPMENT
ADMINISTRATION (ONERA)
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Mathematical theories of acoustics are applied to the propagation of plane, circular, and multidimensional sound waves in homogeneous and inhomogeneous media in motion and at rest. The Mach wave train of a supersonic aircraft in calm and moving air is calculated in detail, taking shock wave propagation and sonic boom prediction into consideration. The structure of nonlinear wave trains is mathematically defined at $O(c_0\epsilon^2)$ perturbation velocity and $O(c_v\epsilon^3)$ entropy perturbation where c_0 is the speed of sound, c_v the velocity of sound in vacuum, and ϵ the order of magnitude of the perturbations. The acoustic effect of the fuselage and of various wing planforms on the ambient air, as a function of the laws of thickness and stress, is calculated. The sonic boom is calculated, with allowance for the dissipation phenomena, and modified equations of motion in canonical Burgers form, are given, using $x = 0$ as the Mach wave associated with supersonic flight. The wake downstream of the aircraft is minimized in the proximity sound field but not defined for the far field.

NOTE

These "Course Notes" have been hastily assembled, with entire disregard of presentation, and mainly have the purpose of rapidly furnishing the student with a working document, giving methods, subject matter, and notations of the oral Course. This text, made available to the students, goes far beyond the frame of the topics of the Course itself which, by necessity, is limited in time. This has been done intentionally so as to give the student a working document that might be useful in some questions closely connected with the Course itself, since it uses methods and notations of the Course. It is obvious that acoustics is a very extensive discipline and that numerous important problems have not been even touched here, even among those that are intimately connected with the subject treated. Finally, the author asks the indulgence of the reader for imperfections of material and presentation and also for all errors committed. The basic aim of producing a sufficiently ample document, in as short a time as possible, did not permit careful editing. We wish to express our thanks to the potential reader and ask him to draw our attention to any possible errors.

THEORETICAL STUDY OF THE PROPAGATION OF SOUND. APPLICATION TO ANTICIPATION OF THE SONIC BOOM PRODUCED BY SUPERSONIC FLIGHT

*/II

J.P. Guiraud

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CHAPTER I

PROPAGATION OF SOUND

1.1 Equations of Acoustics

1.1.1 General Principles

The initial equations are those of the dynamics of compressible fluids

$$\left\{\begin{aligned} &\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho \mathbb{V}) = 0, && a) \\ &\frac{\partial \rho\mathbb{V}}{\partial t} + \nabla\cdot\left(\rho\mathbb{V}\mathbb{V} + p\,\mathbb{I} + \mathbb{T}\right) + \rho\,g = 0, && b) \quad (1) \\ &\frac{\partial \rho(e + V^2/2)}{\partial t} + \nabla\cdot\left\{\rho\left(h + \frac{V^2}{2}\right)\mathbb{V} + \mathbb{Q} + \mathbb{T}\cdot\mathbb{V}\right\} + \rho\, g\cdot\mathbb{V} = 0, && c) \end{aligned}\right.$$

with the conventional notations (g = acceleration of gravity).

Schematization of the Ideal Gas

$$\mathbb{T} = 0 \qquad \mathbb{Q} = 0 \;. \tag{2}$$

Navier-Fourier Schematization

$$\left\{\begin{aligned} &\mathbb{T} = -2\mu\;\mathbb{D} - \left(\mu_v - \frac{2}{3}\mu\right)\mathbb{I}\;\operatorname{div}\mathbb{V}, \\ &\mathbb{Q} = -k \quad T, \end{aligned}\right. \tag{3}$$

where μ_v is the coefficient of viscosity of volume, $\mathbb{D}$ is the component tensor, $D_{ij} = \frac{1}{2}\left(\frac{\partial V_i}{\partial x_j} + \frac{\partial V_j}{\partial x_i}\right)$, and the other notations are conventional.

Equation of state of the gas. It is sufficient to know the function $s(e,\rho)$ where s is the specific entropy; then, T, p, h can be found from

$$ds = \frac{de}{T} - \frac{p}{\rho^2 T}\,d\rho \;, \qquad h = e + \rho^{-1}p \;, \tag{4}$$

but it is more convenient to make use of

$$p = g(\tau, s) \qquad \tau = 1/\rho \tag{5}$$

with the H.Weyl conditions /2

$$g_\tau < 0 \quad , \quad g_s > 0 \quad , \quad g_{\tau\tau} > 0 \, . \tag{6}$$

We pose

$$-\tau^2 g_\tau = c^2 \tag{7}$$

and state that c is the "celerity" or speed of sound; the reason for using this special denotation will be given later in the text.

Problem 1: Starting from $s = c_v\{\log(e\rho^{-(\gamma-1)}) + \text{const}\}$, reconstruct the ideal gas laws.

Demonstrate that the stability of the device shown here requires that $g_T < 0$.

Equation of Entropy

$$\frac{\partial \rho s}{\partial t} + \nabla \cdot \left(\rho s \mathbb{V} + \frac{\mathbb{Q}}{T}\right) = \rho \mathcal{D} \, , \tag{8}$$

where $\rho\mathcal{D}$ denotes the power dissipated per unit mass:

$$\begin{cases} (2) \Longrightarrow & \mathcal{D} = 0 \qquad \text{(no dissipation)} \\ (3) \Longrightarrow & \rho\mathcal{D} = \dfrac{2\mu \; \mathbb{D}:\mathbb{D}}{T} + \dfrac{(\mu_v - 2/3\,\mu)(\operatorname{div}\mathbb{V})^2}{T} + \dfrac{k|\nabla T|^2}{T^2} \end{cases} \tag{9}$$

The Second Law of Thermodynamics, for irreversible transformations, yields

$$\mathcal{D} \geq 0 \quad \Longrightarrow \quad \mu \geq 0, \;\; \mu_v - \frac{2}{3}\mu \geq 0, \;\; k \geq 0 \, . \tag{10}$$

Stokes' hypothesis: $\mu_v = \dfrac{2}{3}\mu$. This holds, according to the kinetic theory of gases, for a monatomic gas but is a priori non-valid in all other cases. The numerical value of μ_v is highly controversial.

Formulas and Numerical Values (Ideal Gas):

$$p = \Re\rho T = \frac{R}{\mathfrak{M}}\rho T = (c_p - c_v)\,\rho T = (\gamma - 1)\, c_v \rho T$$

$$e = c_v T, \; h = c_p T$$

$$p = \rho^\gamma \exp \frac{S + \text{const}}{c_v}$$

$$s + \text{const} = c_p \log T - \mathfrak{R} \log p$$
$$s + \text{const} = c_v \log T - \mathfrak{R} \log \rho$$

$$\gamma = \frac{c_p}{c_v} \quad \begin{cases} 5/3 & \text{monatomic gas} \\ 7/5 = 1.4 & \text{biatomic gas (air)} \end{cases}$$

$R = \mathfrak{N}k = 8.314 \times 10^7$ ergs $({}^\circ\text{Kelvin})^{-1}$ universal constant of ideal gases 13

$k = 1.3805$ erg $({}^\circ K)^{-1}$
$\mathfrak{N} = 6.02 \times 10^{23}$, Avogadro number
$\mathfrak{M} = \mathfrak{N} m$ = molecular mass N_2 28 - O_2 32 - Arg 39.9
Air N_2 78.1% - O_2 2% - Arg 0.9% $\mathfrak{M}$ =
Kinetics model theory of rigid spheres (monatomic gas)

$$\mu_v = 0, \ \mu = \frac{5}{16}\left(\frac{\mathfrak{M}RT}{n\mathfrak{N}^2}\right)^{\frac{1}{2}} \left(\frac{2}{\sigma}\right)^2, \ K = \frac{15}{4}\frac{R}{\mathfrak{M}}\mu$$

σ = diameter of the molecules

$$P_r = \frac{\mu c_p}{k} = \frac{2}{3} \text{ for monatomic gas (air} \sim 0.71)$$

Sutherland formula $\dfrac{\mu(T)}{\mu(T_0)} = \left(\dfrac{T}{T_0}\right)^{\frac{1}{2}} \dfrac{1 + T^*/T_0}{1 + T^*/T}$

$$T^* = 120^\circ K \text{ (air)}$$

	O_2	N_2	Air	
μ g/cm^{-1}/sec^{-1}	1.99×10^{-4}	1.72×10^{-4}	1.78×10^{-4}	$T = 288^\circ K$ Sea-level conditions
λ cm	7.38×10^{-6}	7.36×10^{-6}	7.37×10^{-6}	

Below, we will use the schematization of the ideal gas, up to Chapter V.

1.1.2 Equations of Small Perturbations; Acoustic Energy

We studied the small motions, starting from a homogeneous state at rest, characterized by the subscript zero (at $g = 0$).

$$\begin{aligned} p &= p_0 + \rho_0 c_0^2 (\varepsilon \bar{p}_1 + \varepsilon^2 \bar{p}_2 + \cdots), & \rho &= \rho_0 + \rho_0 (\varepsilon \bar{\rho}_1 + \varepsilon^2 \bar{\rho}_2 + \cdots), \\ e &= e_0 + c_0^2 (\varepsilon \bar{e}_1 + \varepsilon^2 \bar{e}_2 + \cdots), & T &= T_0 + \frac{c_0^2}{C_v} (\varepsilon \bar{T}_1 + \varepsilon^2 \bar{T}_2 + \cdots), \\ s &= s_0 + C_v (\varepsilon \bar{s}_1 + \varepsilon^2 \bar{s}_2 + \cdots), & \mathbb{V} &= c_0 (\varepsilon V_1 + \varepsilon^2 V_2 + \cdots), \end{aligned} \quad (11)$$

with $\boxed{\epsilon \ll 1}$. For making the quantities dimensionless, we introduce

$$\vec{x} = L\vec{\bar{x}} \quad , \quad t = \frac{L}{c_0}\bar{t} \quad , \quad \vec{x} = (x, y, z), \text{ou} = (x_1, x_2, x_3)$$

$$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = \frac{1}{L}\underline{\bar{\nabla}} = \left(\frac{1}{L}\frac{\partial}{\partial \bar{x}}, \frac{1}{L}\frac{\partial}{\partial \bar{y}}, \frac{1}{L}\frac{\partial}{\partial \bar{z}}\right),$$

In the equations of motion and the equations of state, we then substitute the coefficients of the various parts of ϵ and equate them to zero. If the dissipation is negligible everywhere and always, i.e., if there is no shock, and if the perturbations vanish at the initial instant, we will have /4

$$\bar{s}_1 = \bar{s}_2 = \dots = \bar{s}_n = \dots = 0 \, ; \tag{12}$$

Proof: In Lagrange variables, $s - s_0$ = const if one particle is followed in its motion and, at the initial instant, $s = s_0 = 0$.

System of Rank One:

$$\frac{\partial \bar{\rho}_1}{\partial \bar{t}} + \underline{\nabla}\cdot\bar{\mathbb{V}}_1 = 0, \quad \frac{\partial \mathbb{V}_1}{\partial t} + \underline{\nabla}\,\bar{p}_1 = 0, \quad \frac{\partial \bar{\rho}_1 + \frac{c_0^2}{e_0}\bar{e}_1}{\partial t} + \frac{h_0}{e_0}\underline{\nabla}\cdot\mathbb{V}_1 = 0, \tag{12}$$

so that the equations of state will yield

$$\bar{p}_1 = \bar{\rho}_1 \quad , \quad \bar{e}_1 = \frac{p_0}{\rho_0 c_0^2}\bar{\rho}_1 = \gamma^{-1}\bar{\rho}_1 \, . \tag{13}$$

By elimination of $\bar{\mathbb{V}}_1$, it will be found that $\bar{p}_1$ satisfies the wave equation

$$\frac{\partial^2 \bar{p}_1}{\partial \bar{t}^2} - \frac{\partial^2 \bar{p}_1}{\partial \bar{x}^2} - \frac{\partial^2 \bar{p}_1}{\partial \bar{y}^2} - \frac{\partial^2 \bar{p}_1}{\partial \bar{z}^2} \equiv \frac{\partial^2 \bar{p}_1}{\partial t^2} - \underline{\Delta}\,\bar{p}_1 = 0 \, . \tag{14}$$

Since the perturbations are canceled before a certain epoch, we can calculate

$$\bar{P}_1(\bar{x}, \bar{t}) = -\int_{-\infty}^{\bar{t}} \bar{p}_1(\bar{x}, \bar{t}')\, d\bar{t}' \tag{15}$$

and obtain

$$\bar{\mathbb{V}}_1 = \underline{\nabla}\,\bar{P}_1 \tag{16}$$

which shows that the velocity vector is derived from a potential. Let us now write the equation of energy up to the order 2:

$$\varepsilon\left\{\frac{\partial}{\partial t}\left(\bar{\rho}_1+\frac{c_0^2}{e_0}\bar{e}_1\right)+\frac{h_0}{e_0}\underline{\nabla}\cdot\bar{\mathbb{V}}_1\right\}+\varepsilon^2\left\{\frac{\partial}{\partial t}\left[\bar{\rho}_2+\frac{c_0^2}{e_0}\bar{e}_2+\frac{c_0^2}{e_0}\left(\frac{\bar{\mathbb{V}}_1^2}{2}+\frac{p_0}{\rho_0 c_0^2}\bar{\rho}_1^2\right)\right]+\right.$$

$$\left.+\underline{\nabla}\cdot\left[\frac{h_0}{e_0}\bar{\mathbb{V}}_2+\left(\bar{\rho}_1+\frac{c_0^2}{e_0}(\bar{e}_1+\bar{p}_1)\right)\bar{\mathbb{V}}_1\right]\right\}+O(\varepsilon^3)=0,$$

It will be found that cancelation of the coefficient of ϵ furnishes nothing new. For the coefficient of ϵ^2, we will use the following relations:

$$\bar{\rho}_2+\frac{c_0^2}{e_0}\bar{e}_2+\frac{c_0^2}{e_0}\left(\frac{\bar{\mathbb{V}}_1^2}{2}+\frac{p_0}{\rho_0 c_0^2}\bar{\rho}_1^2\right)=\frac{h_0}{e_0}\bar{\rho}_2+\frac{c_0^2}{2e_0}\left(\bar{\mathbb{V}}_1^2+\bar{\rho}_1^2\right)$$

$$\frac{h_0}{e_0}\bar{\mathbb{V}}_2+\left\{\bar{\rho}_1+\frac{c_0^2}{e_0}(\bar{e}_1+\bar{p}_1)\right\}\bar{\mathbb{V}}_1=\frac{h_0}{e_0}\left(\bar{\mathbb{V}}_2+\bar{\rho}_1\bar{\mathbb{V}}_1\right)+\frac{c_0^2}{e_0}\bar{p}_1\bar{\mathbb{V}}_1$$

which result from the equations of state and /5

$$\frac{\partial\bar{\rho}_2}{\partial t}+\underline{\nabla}\cdot\left(\bar{\rho}_1\bar{\mathbb{V}}_1\right)=0\quad,$$

which is obtained from the equation of continuity; then, for the equation of energy of rank 2, we obtain

$$\frac{\partial}{\partial t}\left\{\frac{1}{2}\left(\bar{\mathbb{V}}_1^2+\bar{\rho}_1^2\right)\right\}+\nabla\cdot\left(\bar{p}_1\bar{\mathbb{V}}_1\right)=0\,. \tag{17}$$

Interesting result: The equation of energy of rank 2 does not involve quantities of rank 2 and actually is a relation that is identically verified by the solution of rank 1, known as the acoustic solution.

Problem 2: Write out all equations that control the quantities of rank 2. Establish eq.(12) by recurrence reasoning in Euler variables. If eq.(12) occurs, demonstrate that $\bar{P}_n$ can be defined such that $\bar{\mathbb{V}}_n = \underline{\nabla}\bar{P}_n$.

Volumetric Density of Acoustic Energy:

$$\mathcal{E}=\frac{1}{2}\rho_0 c_0^2\left(\bar{\rho}_1^2+\bar{\mathbb{V}}_1^2\right)\,. \tag{18}$$

Surface Density of the Acoustic Energy Flux:

$$\mathbb{W} = \rho_0 c_0^3 \bar{p}_1 \bar{\mathbb{V}}_1 . \tag{19}$$

From now on, let us omit the dimensionless variables.

Theorem 1: The term "acoustic approximation" is used for describing the flow obtained by means of preceding quantities of rank 1. In acoustic approximation, the velocity vector of flow is derived from a potential, i.e.,

$$\mathbb{V} = \nabla \phi , \tag{20a}$$

which proves the wave equation

$$\frac{1}{c_0^2} \frac{\partial^2 \phi}{\partial t^2} - \Delta \phi = 0 . \tag{21}$$

The pressure reads /6

$$p = p_0 - \rho_0 \frac{\partial \phi}{\partial t} , \tag{20b}$$

while the specific mass is

$$\rho = \rho_0 - \frac{\rho_0}{c_0^2} \frac{\partial \phi}{\partial t} , \tag{20c}$$

while the specific entropy is not perturbed. The volumetric density of acoustic energy and the surface density vector of the acoustic energy flux read as follows:

$$\left\{ \begin{aligned} &\mathcal{E} = \frac{1}{2} \rho_0 \left\{ \frac{1}{c_0^2} \left(\frac{\partial \phi}{\partial t} \right)^2 + |\nabla \phi|^2 \right\} , && a) \\ &\mathbb{W} = - \rho_0 \frac{\partial \phi}{\partial t} \nabla \phi . && b) \end{aligned} \right. \tag{22}$$

If Σ denotes a closed surface, bounding a volume $\mathfrak{D}$ which does not contain sound sources, we have

$$\frac{1}{2} \frac{\partial}{\partial t} \iiint_{\mathfrak{D}} \left(\frac{1}{c_0^2} \left(\frac{\partial \phi}{\partial t} \right)^2 + |\nabla \phi|^2 \right) d\mathfrak{x} - \iint_{\Sigma} \frac{\partial \phi}{\partial t} \frac{d\phi}{dn} \, dS = 0 , \tag{23}$$

where the normal derivative $\frac{d}{dn}$ is evaluated in a direction pointing toward the exterior of $\mathfrak{D}$.

Initial Conditions (Cauchy's Conditions):

$$\begin{cases} \phi(\mathbf{x},0) = F_0(\mathbf{x}), \\ \dfrac{\partial\phi(\mathbf{x},0)}{\partial t} = F_1(\mathbf{x}); \end{cases} \tag{24}$$

If, at the instant t = 0, the perturbations vanish, we will have $F_0 \equiv 0$, $F_1 \equiv 0$. If, at the initial instant, p and $\mathbf{V}$ are given, we will have

$$\nabla F_0(\mathbf{x}) = \mathbf{V}(\mathbf{x}) \qquad , \qquad \frac{p_0 - p}{\rho_0} = F_1(\mathbf{x}) \quad , \text{const} = 0 \; .$$

Limit Conditions (Obstacles):

$$\begin{cases} \dfrac{d\phi}{dn} = 0 & \text{rigid wall, impermeable at rest} \\ \dfrac{d\phi}{dn} = w_n & \text{rigid wall, in known motion} \\ \dfrac{d\phi}{dn} = K\dfrac{\partial\phi}{\partial t} & \text{porous wall, } K > 0, \\ \dfrac{d\phi}{dn} - K\dfrac{\partial\phi}{\partial t} = w_n & \text{porous wall, in motion.} \end{cases} \tag{25}$$

/7

Singularity Conditions

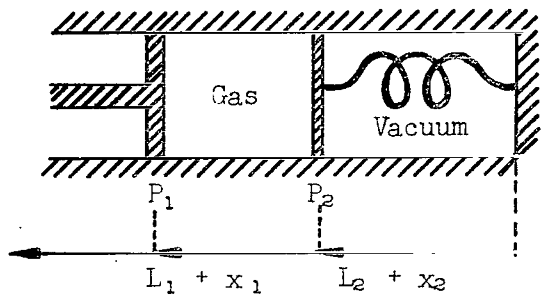
At certain points, along certain lines or along certain surfaces, Φ assumes a given singular behavior. Example:

$$\lim_{|\mathbf{x}| \to 0} \nabla\phi(\mathbf{x},t) = \frac{1}{4\pi}\frac{\mathbf{x}}{|\mathbf{x}|}, \tag{26}$$

which is a source of unit output.

Problem 3: It is assumed that only a certain space variable x exists, so that the most general solution of the wave equation will be $\phi_1(x - c_0 t) + \phi_2(x + c_0 t)$. Let us consider the layout of the accompanying diagram: Here L_1 and L_2 denote the lengths fixing the respective positions of the pistons when the assembly is at rest, with the gas being in the state p_0, ρ_0, c_0. The additional force exerted by the spring is $F_2 = Kx_2$; the piston P_1 has an imposed motion $x_1 = \epsilon L_1 \cos \omega t$; the unit of P_2 + spring behaves as though P_2 had a mass M_2 from the viewpoint of inertia effects; the cross-sectional area of the tube is denoted by S. It is assumed that $\epsilon \ll 1$. Demonstrate that the condition, to be written for the piston P_2, is as follows:

$$\begin{cases} M_2 \dfrac{d^2 x_2}{dt^2} + K x_2 = \rho_0 S' \dfrac{\partial \phi(L_2, t)}{\partial t} \\ \dfrac{d x_2}{dt} = \dfrac{\partial \phi(L_2, t)}{\partial x} \end{cases}$$



while the condition to be written for P_1 is

$$\frac{dx_1}{dt} = \frac{\partial \phi(L_1, t)}{\partial x}.$$

Next, determine φ and x_2.

1.1.3 Inequality of Energy. Uniqueness /8

Let us pose

$$\begin{cases} x_\alpha = (x_1, x_2, x_3, x_0 = ct) = (x_i, x_0) = (\vec{x}, x_0) \\ \mathcal{E}_\alpha = (\mathcal{E}_i, \mathcal{E}_4) \qquad \mathcal{E}_i = -\dfrac{\partial \phi}{\partial x_0} \dfrac{\partial \phi}{\partial x_i}, \quad \mathcal{E}_0 = \dfrac{1}{2}\left(\dfrac{\partial \phi}{\partial x_0}\right)^2 + \dfrac{1}{2}\sum_{i=1}^{3}\left(\dfrac{\partial \phi}{\partial x_i}\right)^2 \end{cases} \tag{27}$$

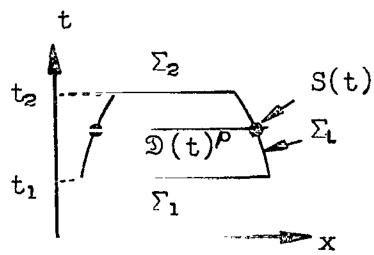
so that eq.(23) is written as

$$\frac{\partial \mathcal{E}_\alpha}{\partial x_\alpha} = 0, \qquad \text{(summation)}. \tag{28}$$

Let Σ be a three-dimensional hypersurface in a four-dimensional space, bounding a hypervolume $\mathfrak{D}$; let n_α be the direction cosines of the normal exterior to Σ, so that the relation (28) will read

$$\iiint_\Sigma n_\alpha \mathcal{E}_\alpha \, dS = 0. \tag{29}$$

Let us apply this formula to the case of the diagram given here: The lateral surface Σ_L is obtained by a timewise displacement of the (ordinary)



surface S(t), limiting the (ordinary) volume $\mathfrak{D}$(t), and then its equation is taken in the form

$$\psi(x_1, x_2, x_3) + x_0 \equiv \psi(x_1, x_2, x_3) + c_0 t = 0, \tag{30}$$

denoting by $\mathfrak{n}$ the unit vector normal to S(t) in ordinary space, pointing toward the exterior of $\mathfrak{D}$(t); we then note

$$\nabla \psi = -\frac{c_0}{w_n}\mathfrak{n}, \tag{31}$$

by definition of the normal rate of displacement w_n d S(t); finally, we pose

$$E(t) = \frac{1}{2}\iiint_{\mathfrak{D}(t)} \rho_0 \left\{ \frac{1}{c_0^2}\left(\frac{\partial \phi}{\partial t}\right)^2 + |\nabla \phi|^2 \right\} dx, \tag{32}$$

for the acoustic energy contained, at the instant t, within the volume $\mathfrak{D}$(t). /9
From this, we obtain the equality of energy

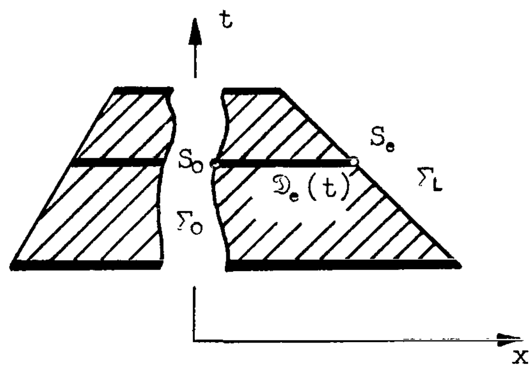
$$E(t_2) - E(t_1) - \frac{1}{2}\rho_0 \iiint_{\Sigma_L} \left\{ \left| \nabla\phi + \frac{\mathfrak{n}}{w_n}\frac{\partial\phi}{\partial t} \right|^2 + \left(1 - \frac{c_0^2}{w_n^2}\right)\frac{1}{c_0^2}\left(\frac{\partial\phi}{\partial t}\right)^2 \right\} \frac{w_n\, dS}{\sqrt{c_0^2 + w_n^2}} = 0 \tag{33}$$

where dS denotes the area element of Σ_L in the space x_α.

Theorem 2 (Energy inequality): The acoustic energy contained within a domain $\mathfrak{D}$(t), whose boundary is everywhere displaced in the sense of a contraction at a velocity equal or superior to c_0, can only diminish or remain constant. This energy decreases strictly if, on a finite segment of the boundary where ϕ does not vanish, a velocity of normal displacement greater than c_0 is present.

Theorem 3 (Uniqueness): Let us assume that ϕ and $\frac{\partial\phi}{\partial t}$ vanish identically at the instant t = 0 in the sphere with the center x and the radius r; let us assume, in addition, that ϕ at each instant proves the wave equation in the sphere with the center x and the radius $r - c_0 t$ (>0), while $\phi \equiv 0$ at the instant t in the same sphere.

Let us now consider the case in which limiting conditions must be taken into consideration. Let us discuss the exterior problem, i.e., the problem of determining the acoustic field at the exterior of an obstacle. Let us apply eq.(29) to the case of the diagram given here. Let us denote, by $\mathfrak{D}_e(t)$, the



domain considered as being exterior to the obstacle $S_0(t)$ and let $S_e(t)$ be the exterior surface of $\mathfrak{D}_e(t)$. Let us denote by w_n the velocity of normal displacement of S_0 as well as of S_e (> 0 if there is an increment); eq.(33) must then be replaced by

/10

$$E(t_2) - E(t_1) + \frac{1}{2}\rho_0 \int_{t_1}^{t_2} dt \iint_{S_0(t)} \left\{ \left[|\nabla_T \phi|^2 + \left(\frac{d\phi}{dn}\right)^2 + \frac{1}{c_0^2}\left(\frac{\partial\phi}{\partial t}\right)^2 \right] w_n + 2\frac{\partial\phi}{\partial t}\frac{d\phi}{dn} \right\} dS_0$$

$$= \frac{1}{2} \iiint_{\Sigma_L} \left\{ \left| \nabla\phi + \frac{\mathfrak{n}}{w_n}\frac{\partial\phi}{\partial t} \right|^2 + \left(1 - \frac{c_0^2}{w_n^2}\right)\frac{1}{c_0^2}\left(\frac{\partial\phi}{\partial t}\right)^2 \right\} \frac{w_n\, dS}{\sqrt{c_0^2 + w_n^2}} \qquad (34)$$

and, if $S_e(t)$ contracts with supersonic velocity,

$$E(t_2) \leq E(t_1) - \frac{1}{2}\rho_0 \int_{t_1}^{t_2} dt \iint_{S_0(t)} \left\{ \left[|\nabla_T \phi|^2 + \left(\frac{d\phi}{dn}\right)^2 + \frac{1}{c_0^2}\left(\frac{\partial\phi}{\partial t}\right)^2 \right] w_n + 2\frac{\partial\phi}{\partial t}\frac{d\phi}{dn} \right\} dS_0. \qquad (35$$

Important note: If the consequences of linearization are pushed to the extreme, it is necessary to set $w_n = 0$ in eq.(35). This results from

$$\left[|\nabla_T \phi|^2 + \cdots \right] |W_M| = O(c_0^3 \varepsilon^3)$$

$$\left| \frac{\partial \phi}{\partial t} \frac{d\phi}{dn} \right| = O(c_0^3 \varepsilon^2)$$

or else from the principle of transfer of the limiting conditions: Each time that, in acoustics, a limiting condition must be written for a surface whose velocity of normal displacement is of the same order as the velocity of perturbation of the gas, this condition must be applied to the mean-position fictive surface of the real surface.

Theorem 4: Let us consider a domain $\mathfrak{D}(t)$ of three-dimensional space which, on one hand, is given by a surface exhibiting limit conditions, i.e., $S_0(t)$, and on the other hand by a surface which contracts at a velocity which is everywhere supersonic. Let us assume that ϕ is the velocity potential in $\mathfrak{D}(t)$ ($t_1 \leq t \leq \tau_2$) of an acoustic field which obeys the principle of limiting condition transfer; then, the acoustic energy $E(t)$, contained in the domain $\mathfrak{D}(t)$, proves the energy inequality ($\frac{d}{dn}$ normal exterior value):

$$E(t_2) \leq E(t_1) - \int_{t_1}^{t_2} dt \iint_{S_0(t)} \rho_0 \frac{\partial \phi}{\partial t} \frac{d\phi}{dn} \, dS_0 \qquad (36)$$

Theorem 5 (Uniqueness): Let us consider two acoustic fields with velocity potentials ϕ_1 and ϕ_2, proving /11

$$\phi_1 = \phi_2 \qquad \frac{\partial \phi_1}{\partial t} = \frac{\partial \phi_2}{\partial t} \qquad (37)$$

at $t = 0$ within a domain $\mathfrak{D}(0)$; let us assume that, on the surfaces with limiting conditions, relative to the transfer principle, we have

$$\alpha \frac{d\, \phi_1 - \phi_2}{dn} - \beta \frac{\partial\, \phi_1 - \phi_2}{\partial t} = 0 \qquad \alpha, \beta \geq 0 \qquad (38)$$

so that

$$\phi_1 \equiv \phi_2 \qquad (39)$$

is obtained for each instant $t > 0$ and each domain $\mathfrak{D}(t)$, proving the following conditions: Since the functions ϕ_1 and ϕ_2 prove the wave equation in $\mathfrak{D}(\tau)$, $0 \leq \tau \leq t$, the domain $\mathfrak{D}(\tau)$ is bounded, at the interior, by the surface or surfaces with limiting conditions, and is bounded, at the exterior, by a surface which contracts everywhere at a velocity equal or superior to c_0.

The proof is furnished by eqs.(36) and (38).

1.2 First Solutions of the Wave Equation

1.2.1 Plane Waves

Let ω be a unit vector, so that

$$\phi(\vec{x}, t) = F(\omega \cdot x - c_0 t), \tag{1}$$

is a solution of the wave equation if $F(z)$ is twice continuously derivable. It is said that a "plane wave" propagates in the direction of the vector ω. We then calculate

$$\mathbb{V} = \omega\, F'(\omega x - c_0 t) \tag{2}$$

and so on.

Theorem 6: In an acoustic field, propagating by plane waves in the direction ω, we have /12

$$\begin{cases} p = p_0 + \rho_0 c_0\, \omega \cdot \mathbb{V}, \\ \mathcal{E} = \dfrac{(p - p_0)^2}{\rho_0 c_0^2}, \\ \mathbb{W} = c_0\, \omega\, \mathcal{E}. \end{cases} \tag{3}$$

Harmonic plane wave:

$$F(z) = A\, \mathcal{R}e\; e^{-ikz + i\theta}$$

where k is the wave number, and $\omega = kc_0$ is the pulsation.

We then find

$$\begin{cases} \mathcal{E} = \boxed{\dfrac{\rho_0 k^2 A^2}{2}} - \dfrac{\rho_0 k^2 A^2}{2} \cos\{2k(\omega \cdot x - c_0 t) - 2\theta\}, \\ \mathbb{W} = \boxed{\dfrac{\rho_0 k^2 c_0 A^2}{2}} - \dfrac{\rho_0 k^2 c_0 A^2}{2}\, \omega \cos\{2k(\omega \cdot x - c_0 t) - 2\theta\}. \end{cases} \tag{4}$$

When speaking of $\mathfrak{E}$ and $\mathbf{W}$ for a harmonic plane wave, it is frequently convenient to retain only the permanent portions (boxed terms).

Important note: In a plane wave, whether harmonic or not, equipartition exists between the energy of kinetic origin and the energy of thermal origin, i.e.,

$$\frac{1}{c_0^2}\left(\frac{\partial \phi}{\partial t}\right)^2 = |\nabla \phi|^2 . \tag{5}$$

1.2.2 Spherical Waves

The equation

$$\phi(\boldsymbol{x}, t) = \iint_{\Omega} A(\boldsymbol{\omega})\, F(\boldsymbol{\omega}\cdot\boldsymbol{x} - c_0 t)\, d\boldsymbol{\omega} \tag{6}$$

where r is a sphere of radius 1, represents a solution of the wave equation. /13

Let us use $A(\boldsymbol{\omega}) = \frac{1}{4n}$ and let $G'(z) = F(z)$, so that

$$\frac{1}{4\pi}\iint_{\Omega} F(\boldsymbol{\omega}\cdot\boldsymbol{x} - c_0 t)\, d\boldsymbol{\omega} = \frac{1}{2}\int_0^{\pi} F(r\cos\theta - c_0 t)\sin\theta\, d\theta$$
$$= \frac{G(r - c_0 t) - G(-r - c_0 t)}{2r} \tag{7}$$

if we pose $|\boldsymbol{x}| = r$.

If the function Φ depends only on $r = |\boldsymbol{x}|$ and on t but not on the direction of the vector $\boldsymbol{x}$, so that

$$\frac{1}{c_0^2}\frac{\partial^2 \phi}{\partial t^2} - \Delta\phi = \frac{\partial^2 r\phi}{c_0^2\,\partial t^2} - \frac{\partial^2 r\phi}{\partial r^2} , \tag{8}$$

then, the most general solution of the wave equation having spherical symmetry will be

$$\phi(r, t) = \frac{F_1(r - c_0 t) + F_2(r - c_0 t)}{r} . \tag{9}$$

Radiant spherical wave (source):

$$\phi_R = -\frac{1}{4\pi\rho_0}\,\frac{\mathfrak{M}(t - r/c_0)}{r}\,. \tag{10}$$

The mass flux, emerging from a sphere of radius r, tends toward $\mathfrak{M}(t)$ if $r \to 0$.

Antiradiant spherical wave:

$$\phi_{AR} = -\frac{1}{4\pi\rho_0}\,\frac{\mathfrak{M}(t + r/c_0)}{r}\,, \tag{11}$$

having the same output property.

Stationary spherical wave:

$$\phi_S = \frac{1}{4\pi\rho_0}\,\frac{\mathfrak{M}(t + r/c_0) - \mathfrak{M}(t - r/c_0)}{r}\,, \tag{12}$$

which has zero output through a sphere of zero radius.

Acoustic energy: /14

$$\text{R.S.W.}\left\{\begin{aligned} \mathcal{E} &= \frac{1}{32\pi^2\rho_0 c_0^2}\,\frac{\mathfrak{M}'^2 + \left(\mathfrak{M}' + \frac{c_0\mathfrak{M}}{r}\right)^2}{r^2}\,,\\ W &= \frac{1}{16\pi^2\rho_0 c_0}\,\frac{\mathfrak{M}'\left(\mathfrak{M}' + \frac{c_0\mathfrak{M}}{r}\right)}{r^2}\,\frac{\mathcal{X}}{r}\,, \end{aligned}\right. \tag{13}$$

$$\text{A.S.W.}\left\{\begin{aligned} \mathcal{E} &= \frac{1}{32\pi^2\rho_0 c_0^2}\,\frac{\mathfrak{M}'^2 + \left(\mathfrak{M}' - \frac{c_0\mathfrak{M}}{r}\right)^2}{r^2}\\ W &= -\frac{1}{16\pi^2\rho_0 c_0}\,\frac{\mathfrak{M}'\left(\mathfrak{M}' - \frac{c_0\mathfrak{M}}{r}\right)}{r^2}\,\frac{\mathcal{X}}{r}\,. \end{aligned}\right.$$

Posing

$$\mathcal{X} = \omega r\,, \tag{14}$$

the relation of the plane wave

$$\mathfrak{W} = \pm \; \xi \, c_0 \, \varpi \qquad \begin{array}{l} + \longleftrightarrow \text{R.S.W.} \\ - \longleftrightarrow \text{A.S.W.} \end{array} \tag{15}$$

is approximately proved, under the condition

$$\left| \frac{c_0 \mathfrak{M}}{r} \right| \ll \left| \mathfrak{M}' \right| , \tag{16}$$

and under the same standard as the equipartition.

Harmonic spherical wave:

$$\mathfrak{M}(t) = \mathfrak{M}_0 \, \mathcal{R}e \; e^{i(\omega t + \theta)} \; ; \qquad \omega = c_0 k \tag{17}$$

Here, we leave the task of explicit definition to the reader. Let us merely mention that, for a radiant wave, we have

$$\left\{ \begin{array}{l} \xi = \boxed{\dfrac{\mathfrak{M}_0^2 \, \omega^2}{32 \pi^2 \rho_0 c_0^2 r^2} \left(1 + \dfrac{1}{2 k^2 r^2} \right)} + \text{Periodic} \\[2ex] \mathfrak{W} = \boxed{\dfrac{\mathfrak{M}_0^2 \, \omega^2}{32 \pi^2 \rho_0 c_0 r^2} \, \varpi} + \text{Periodic} \end{array} \right. \tag{18}$$

(do not confuse ω and ϖ)

Thus, we can make the following statement: A harmonic spherical wave behaves locally and approximately like a plane wave for

$$k r \gg 1 \, . \tag{19}$$

The energy flux of a harmonic spherical wave, either radiant or antiradiant, emerging from a sphere of radius r is, in mean time, independent of r. /15

Problem 4: Demonstrate that, if a source discharges during a limited interval of time, it follows that, at each ulterior instant,

$$\int_0^{\infty} r (p - p_0) \, dr = 0 \, , \tag{20}$$

and, at each point,

$$\int_{-\infty}^{\infty} (p - p_0)\, dt = 0 \; . \tag{21}$$

Deduce from this, if a spherical wave of limited extent in time is observed at a given point, this wave necessarily comprises compression phases and expansion phases.

1.2.3 Cylindrical Waves

The quantity $F(x \cos \theta + y \sin \theta = c_0 t)$ is an independent solution of z, so that

$$\phi(x, y, t) = \int_0^{2\pi} A(\theta)\, F(x \cos \theta + y \sin \theta - c_0 t)\, d\theta \; , \tag{22}$$

is a solution of the wave equation in two-dimensional space. We use $A(\theta) = \frac{1}{2\pi}$, $\int_0 F(z) = -\mathfrak{M}\left(-\frac{z}{c_0}\right)$, so as to obtain, with

$$\phi(r, t) = -\frac{1}{2\pi \rho_0} \int_0^{2\pi} \mathfrak{M}\left(t - \frac{r}{c_0} \cos \theta\right) d\theta \tag{23}$$

a solution which depends only on $r = \sqrt{x^2 + y^2}$, i.e., a cylindrical wave.

By change of variable, we obtain /16

$$\phi_S(r, t) = -\frac{1}{2\pi \rho_0} \int_{t - r/c_0}^{t + r/c_0} \frac{\mathfrak{M}(\tau)}{\sqrt{\frac{r^2}{c_0^2} - (\tau - t)^2}}\, d\tau \tag{24}$$

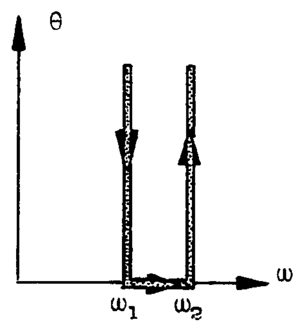
thus yielding a stationary cylindrical wave. We then wish to form radiant and antiradiant cylindrical waves. For this, we superpose plane waves of a complex propagation direction. Thus,

$$\phi(x, y, t) = -\frac{1}{2\pi \rho_0} \mathrm{Re} \int_0^{\infty} \mathfrak{M}\left(t - \frac{x}{c_0} \coth \theta + i \frac{y}{c_0} \sinh \theta\right) d\theta \tag{25}$$

is a solution of the wave equation if $\mathfrak{M}$ is analytical. With $x + iy = re^{i\omega}$, we obtain

$$\phi(r,t) = -\frac{1}{2\pi\rho_0} \mathcal{R}e \int_0^\infty \mathfrak{M}\left\{t - \frac{r}{c_0}\cos(\omega + i\theta)\right\} d\theta \tag{26}$$

since we will demonstrate that ϕ does not depend on ω. For simplification, let us assume that $\mathfrak{M}\left\{t - \frac{r}{c_0}\cos(\omega + i\theta)\right\} \to 0$ rather rapidly, as soon as $\theta \to \infty$. Cauchy's theorem, when applied to the contour of the accompanying diagram,



will yield

$$i\int_0^\infty \left[\mathfrak{M}\left\{t - \frac{r}{c_0}\cos(\omega_2 + i\theta)\right\} - \mathfrak{M}\left\{t - \frac{r}{c_0}\cos(\omega_1 + i\theta)\right\}\right] d\theta$$
$$+ \int_{\omega_1}^{\omega_2} \mathfrak{M}\left(t - \frac{r}{c_0}\cos\omega\right) d\omega = 0 ,$$

so that the second term of eq.(26) does not depend on ω if $\mathfrak{M}(z)$ is real for a real z, which we assume to be the case here.

Then,

$$\phi(r,t) = -\frac{1}{2\pi\rho_0} \int_0^\infty \mathfrak{M}\left(t - \frac{r}{c_0}\,\mathrm{ch}\,\theta\right) d\theta , \tag{27}$$

and, on change of variable, we have

$$\phi_R(r,t) = -\frac{1}{2\pi\rho_0}\int_{-\infty}^{t-r/c_0} \frac{\mathfrak{M}(\tau)}{\sqrt{(t-\tau)^2 - \frac{r^2}{c_0^2}}}\, d\tau . \tag{28}$$

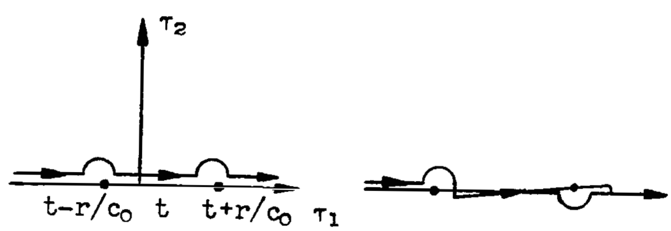
We leave to the reader the task of demonstrating that, by changing from minus to plus in eq.(27), the following is obtained: /17

$$\phi_{AR}(r,t) = -\frac{1}{2\pi\rho_0}\int_{t+r/c_0}^{\infty} \frac{\mathfrak{M}(\tau)}{\sqrt{(\tau-t)^2 - \frac{r^2}{c_0^2}}}\, d\tau . \tag{29}$$

If the function $\mathfrak{M}(\tau)$ is analytical, we can group eqs.(24), (28), and (29) into a same formulation, by posing

$$\phi(r,t) = -\frac{1}{2\pi\rho_0}\int_L \frac{\mathfrak{M}(\tau)}{\sqrt{(\tau-t)^2 - \frac{r^2}{c_0^2}}}\, d\tau , \tag{30}$$

where L denotes a convenient path in the complex plane $\tau = \tau_1 + L\tau_2$. It is



Path L_1 Path L_2

agreed that the argument of the radical is zero for $\tau = -\infty + 0i$.

$$\begin{cases} \text{Path } L_1 & \phi = \phi_R - \phi_{AR} + i\,\phi_S , \\ \text{Path } L_2 & \phi = \phi_R + \phi_{AR} + i\,\phi_S . \end{cases} \tag{31}$$

Naturally, the value of the second term in eq.(30) does not depend on the radius of the semicircular indentations.

<u>Problem 5</u>: By direct calculation, prove that one has

$$\left(\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}-\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}\right)\frac{1}{\sqrt{c_0^2t^2-r^2}}=0\,. \tag{32}$$

Demonstrate that this calculation is made obsolete if it is known that

$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}+\frac{\partial^2}{\partial z^2}\right)\frac{1}{(x^2+y^2+z^2)^{1/2}}=0\,. \tag{33}$$

It is stated that ϕ_R, ϕ_{AR}, and ϕ_S which are, respectively, defined by eqs.(28), (29), and (24), are cylindrical waves which are, respectively, radiant, antiradiant, and stationary. If $\mathfrak{M}(\tau)$ is analytical and vanishes sufficiently rapidly at infinity, it is certain that, in accordance with eqs.(30) and (31), these represent solutions of the wave equation. Let us note that the analyticity has the effect to exclude $\tau = t \pm r/c_0$ where the differentiation under the summation sign cannot be made. In fact, this hypothesis of analyticity is much too strict. It is sufficient that $\mathfrak{M}(t)$ be twice continuously differentiable to permit a double differentiation under the summation sign, by means of one or the other of two technical artifices: partial finite integration (Hadamard) and integration in the Riemann-Liouville sense (M.Riesz). We will return to this point later. /18

Problem 6: Establish the relations

$$\left\{\begin{array}{l}\displaystyle\frac{1}{2}\int_{-\infty}^{\infty}\frac{\mathfrak{M}\left(t-\frac{\sqrt{r^2+z^2}}{c_0}\right)}{\sqrt{r^2+z^2}}\,dz=-\int_{-\infty}^{t-r/c_0}\frac{\mathfrak{M}(\tau)}{\sqrt{(t-\tau)^2-r^2/c_0^2}}\,d\tau\,,\\[2ex]\displaystyle\frac{1}{2}\int_{-\infty}^{\infty}\frac{\mathfrak{M}\left(t+\frac{\sqrt{r^2+z^2}}{c_0}\right)}{\sqrt{r^2+z^2}}\,dz=\int_{t+r/c_0}^{\infty}\frac{\mathfrak{M}(\tau)}{\sqrt{(\tau-t)^2-r^2/c_0^2}}\,d\tau\,.\end{array}\right. \tag{34}$$

and interpret them.

To understand its physical significance, let us define how a cylindrical wave of one or the other type will behave at $r \to 0$. We have

$$\begin{cases} \phi_R = -\dfrac{1}{2\pi\rho_0}\displaystyle\int_{-\infty}^{t-r/c_0} \mathfrak{M}'(\tau)\,\mathrm{Arg\,coth}\,\dfrac{c_0(t-\tau)}{r}\,d\tau, & a) \\ \phi_{AR} = \dfrac{1}{2\pi\rho_0}\displaystyle\int_{t+r/c_0}^{\infty} \mathfrak{M}'(\tau)\,\mathrm{Arg\,coth}\,\dfrac{c_0(\tau-t)}{r}\,d\tau, & b) \end{cases} \tag{35}$$

whereas

$$\mathrm{Arg\,coth}\,\frac{c_0(t-\tau)}{r} = \mathrm{Log}\,\frac{1}{r} + \mathrm{Log}\left[2c_0(t-\tau)\right] + \frac{r^2}{c_0^2(t-\tau)^2}\,M(c_0t, c_0\tau, r) \tag{36}$$

where M remains bounded in the useful interval by eq.(35a), with an analogous /19 formula for eq.(35b). From this, it follows that

$$\begin{aligned} \phi_R &= -\frac{\mathfrak{M}(t-r/c_0)}{2\pi\rho_0}\,\mathrm{Log}\,\frac{1}{r} - \frac{1}{2\pi\rho_0}\int_{-\infty}^{t-r/c_0} \mathfrak{M}'(\tau)\,\mathrm{Log}\left[2c_0(t-\tau)\right]d\tau + r\,\mathfrak{N}_R(r, c_0t) \\ \phi_{AR} &= -\frac{\mathfrak{M}(t+r/c_0)}{2\pi\rho_0}\,\mathrm{Log}\,\frac{1}{r} + \frac{1}{2\pi\rho_0}\int_{t+r/c_0}^{\infty} \mathfrak{M}'(\tau)\,\mathrm{Log}\left[2c_0(\tau-t)\right]d\tau + r\,\mathfrak{N}_{AR}(r, c_0t) \end{aligned} \tag{37}$$

where $\mathfrak{N}$ always remains bounded when r tends to zero. In fact, we wish to know more precisely the behavior of $\frac{\partial\varphi}{\partial r}$. For this, we start from eq.(35) which, by differentiation under the summation sign and then by partial integration, leads to

$$\begin{cases} \dfrac{\partial\phi_R}{\partial r} = \dfrac{1}{2\pi\rho_0 r}\displaystyle\int_{-\infty}^{t-r/c_0} \mathfrak{M}''(\tau)\sqrt{(t-\tau)^2 - \dfrac{r^2}{c_0^2}}\;d\tau \\ \dfrac{\partial\phi_{AR}}{\partial r} = \dfrac{1}{2\pi\rho_0 r}\displaystyle\int_{t+r/c_0}^{\infty} \mathfrak{M}''(\tau)\sqrt{(\tau-t)^2 - \dfrac{r^2}{c_0^2}}\;d\tau \end{cases} \tag{38}$$

Then, we note that

$$\frac{\tau\sqrt{\tau^2 - \tau_0^2} - \tau^2}{\tau_0^2} \text{ remains bounded for } \tau_0 \leq \tau \leq \infty$$

which permits obtaining, as above, in view of

$$\int_{-\infty}^{t-r/c_0} (t-\tau)\, \mathfrak{M}''(\tau)\, d\tau = \mathfrak{M}(t - r/c_0) + \frac{r}{c_0}\mathfrak{M}'(t - r/c_0)$$

$$\int_{t+r/c_0}^{\infty} (\tau - t)\, \mathfrak{M}''(\tau)\, d\tau = \mathfrak{M}(t + r/c_0) - \frac{r}{c_0}\mathfrak{M}'(t + r/c_0).$$

with, to terminate,

$$\left\{\begin{aligned} \frac{\partial\phi_R}{\partial r} &= \frac{\mathfrak{M}(t - r/c_0)}{2\pi\rho_0 r} + Q_R(r, c_0 t)\,\mathrm{Log}\,\frac{1}{r}, \\ \frac{\partial\phi_{AR}}{\partial r} &= \frac{\mathfrak{M}(t + r/c_0)}{2\pi\rho_0 r} + Q_{AR}(r, c_0 t)\,\mathrm{Log}\,\frac{1}{r}, \end{aligned}\right. \tag{39}$$

where the quantities Q always remain bounded when $r \to 0$.

Thus, the amount of mass emerging from a cylinder of evanescent radius and unit height is equal to $\mathfrak{M}(t)$, so that the radiant and antiradiant spherical waves are radiant or antiradiant acoustic fields produced by a source uniformly distributed over the axis of symmetry. For a stationary wave, the emission is zero. /20

Let us study the case of a pulsating source, i.e.,

$$\mathfrak{M}(t) = \mathfrak{M}_0\, e^{i\omega t}, \tag{40}$$

from which we obtain

$$\phi(r,t) = e^{i\omega t}\, u(r,t), \qquad \omega = kc_0 \tag{41}$$

and

$$\begin{cases} u_R = -\dfrac{\mathfrak{M}_0}{2\pi\rho_0}\displaystyle\int_0^\infty e^{-i\omega\frac{r}{c_0}\coth\theta}\,d\theta = -\dfrac{\mathfrak{M}_0}{4i\rho_0}\,H_0^{(2)}(kr)\,, \\ u_{AR} = -\dfrac{\mathfrak{M}_0}{2\pi\rho_0}\displaystyle\int_0^\infty e^{i\omega\frac{r}{c_0}\coth\theta}\,d\theta = \dfrac{\mathfrak{M}_0}{4i\rho_0}\,H_0^{(1)}(kr)\,, \\ u_S = -\dfrac{\mathfrak{M}_0}{2\pi\rho_0}\displaystyle\int_{-\pi/2}^{\pi/2} e^{i\omega\frac{r}{c_0}\sin\theta}\,d\theta = -\dfrac{\mathfrak{M}_0}{2}\,J_0(kr)\,, \end{cases} \tag{42}$$

with

$$H_0^{(1,2)}(x) = J_0(x) \pm iY_0(x)\,, \qquad \begin{matrix} + \leftrightarrow 1 \\ - \leftrightarrow 2 \end{matrix} \tag{43}$$

where J_0 is a Bessel function, Y_0 a Neumann function, and $H_0^{(1,2)}$ are Hankel functions.

In terminating this Section, we will study the behavior of a cylindrical wave at a great distance from the axis of symmetry. Let us consider a radiant wave and assume

$$\mathfrak{M}(\tau) \equiv 0\,, \quad \text{outside of} \quad 0 \leq \tau \leq \tau_0 = \frac{\ell}{c_0}\,. \tag{44}$$

Let us first assume that /21

$$\frac{c_0 t}{\ell} \to \infty\,, \quad \frac{r}{\ell} \to \infty\,, \quad \frac{c_0 t}{r} > 1\,, \tag{44}$$

We then readily obtain the so-called principal term of the distal asymptotic representation of ϕ_R, i.e.,

$$\phi_R^{(D)} = -\frac{1}{2\pi\rho_0}\int_0^{\tau_0}\mathfrak{M}(\tau)\,d\tau\,\frac{1}{\sqrt{t^2 - \frac{r^2}{c_0^2}}}\,, \tag{45}$$

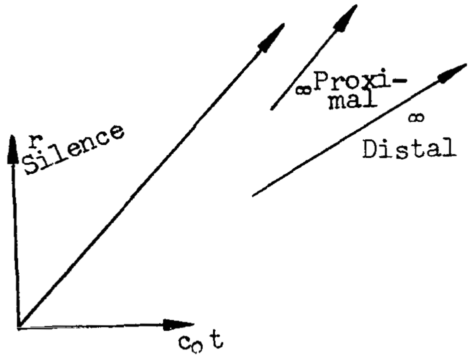
with a relative error of $O\left(\frac{\ell}{r}\right)$. Let us now assume that

$$\frac{c_0 t}{\ell} \to \infty\,, \quad \frac{r}{\ell} \to \infty\,, \quad t - \frac{r}{c_0} = \theta \quad \text{remaining fixed.} \tag{46}$$

By a procedure analogous to the above process, we obtain the so-called principal term of the proximal asymptotic representation of ϕ_R, i.e.,

$$\phi_R^{(P)} = -\frac{1}{2\pi\rho_0\sqrt{\frac{2r}{c_0}}}\int_0^{\mathrm{Inf}(\theta,\tau_0)} \frac{\mathcal{M}(\tau)}{\sqrt{\theta-\tau}}\,d\tau, \tag{47}$$

with a relative error which still is $O\left(\frac{\ell}{r}\right)$. Naturally, it is possible to write, in each case, a complete asymptotic expansion which, incidentally, converges first if $\frac{c_0 t}{r} \geq 1 + \epsilon$ ($\epsilon > 0$) and, secondly, if θ is bounded. We leave



to the reader the task of writing out this expansion. In fact, each of these series can be extended beyond its strict validity domain into a common validity domain. This will show by the existence of connectivity conditions. In fact, let us study the proximal asymptotic behavior, meaning that, under the conditions (46), from $\phi_R^{(D)}$, we find

/22

$$\phi_R^{(P)} \sim -\frac{1}{2\pi\sqrt{\frac{2r}{c_0}}\sqrt{\theta}}\int_0^{\tau_0} \frac{\mathcal{M}(\tau)}{\rho_0}\,d\tau; \tag{48}$$

Let us next investigate the distal asymptotic behavior of $\phi_R^{(P)}$, finding

$$\phi_R^{(P)} \sim -\frac{1}{2\pi\sqrt{\frac{2r}{c_0}}\sqrt{\theta}}\int_0^{\tau_0} \frac{\mathcal{M}(\tau)}{\rho_0}\,d\tau. \tag{49}$$

The identity of the second terms of eqs.(48) and (49) expresses one of the connectivity conditions. In the actual case, it is not a question of a condition but rather of the constatation of an identity; this is so, since we had constructed the asymptotic expansions from an explicit representation of the solution. In numerous cases, it is impossible to obtain the solution of the problem itself, whereas, to make up for this, it is possible to obtain different asymptotic representations of the problem which are valid in various regions. Such representations would contain constants or unknown functions; some of the connectivity conditions permit a determination of these constants or functions while others are reduced, as above, to identities. For further details on these questions, reference is made to P.A.Lagerstrom in his Superior Course of Aerodynamics, 1960 - 1961, published by the Department of Mechanics, Paris.

Problem 7: Establish the relations

$$\mathcal{E}_R^{(P)} = \frac{1}{8\pi^2 \rho_0 c_0 r}\left(\int_0^{\mathrm{Inf}(t-r/c_0,\, \ell)} \frac{m'(\tau)}{\sqrt{t-\frac{r}{c_0}-\tau}}\, d\tau\right)^2 \left\{1+O\left(\frac{\ell}{r}\right)\right\}, \quad a)$$

$$W_R^{(P)} = c_0 \mathcal{E}_R^{(P)} \omega \left\{1+O\left(\frac{\ell}{r}\right)\right\}, \quad b) \tag{50}$$

Next, fully treat the case of an antiradiant wave and then that of a stationary wave. Then, interpret eq.(50b).

1.2.4 Waves in the Distribution Sense /23

Even if the function F is not differentiable, eq.(1) is a distribution solution of the wave equation. It is stated occasionally that this is a weak solution. It is not even necessary that F be a function, it can also be a distribution. For example, $\delta(\omega \cdot x - c_0 t)$ is a solution. Taking $\vec{\omega}$ in the direction of the axis ox, we will obtain

$$\left\langle \left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2}\right)\delta(x-c_0t),\ \psi \right\rangle = \left\langle \delta(x-c_0t),\ \frac{\partial^2\psi}{c_0^2\partial t^2} - \frac{\partial^2\psi}{\partial x^2}\right\rangle = 0. \tag{51}$$

In fact, let us pose

$$c_0t = \xi \qquad x + c_0t = \eta \qquad \psi(x,t) = \bar{\psi}(\xi,\eta)$$

We then have

$$\left\langle \delta(x-c_0t), \frac{\partial^2\psi}{c_0^2\partial t^2} - \frac{\partial^2\psi}{\partial x^2}\right\rangle = -\frac{2}{c_0}\int_{-\infty}^{\infty}\frac{\partial^2\bar{\psi}(0,\eta)}{\partial\eta\,\partial\xi}\,d\eta = -\frac{2}{c_0}\left(\frac{\partial\bar{\psi}(0,\eta)}{\partial\xi}\right)_{-\infty}^{\infty} = 0$$

since ψ vanishes identically in the neighborhood of infinity. The above calculation can be avoided by taking into consideration that the derivatives $\frac{\partial^2}{\partial t^2}$ and $\frac{\partial^2}{\partial x^2}$ of $\delta(x - c_0 t)$ can be calculated by making use of the derivation rule of functions of functions ($\delta(x - c_0 t) = \delta(F)$, $F = x - c_0 t$).

Now, let ϕ be a spherical or cylindrical wave of one or the other of three types. We then have

$$\left\langle \frac{1}{c_0^2}\frac{\partial^2\phi}{\partial t^2} - \Delta\phi, \psi \right\rangle = \left\langle \phi, \frac{1}{c_0^2}\frac{\partial^2\psi}{\partial t^2} - \Delta\psi \right\rangle =$$

$$\lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} dt \int_{\mathfrak{D}_\varepsilon} \phi \left(\frac{1}{c_0^2}\frac{\partial^2\psi}{\partial t^2} - \Delta\psi \right) d\vec{x}$$

where $\mathfrak{D}_\varepsilon$ is the domain characterized by $r > \epsilon$ (of the two-dimensional or three-dimensional type). If $\mathfrak{M}(t)$ is twice continuously differentiable, then ϕ is also twice continuously differentiable in $\mathfrak{D}_\varepsilon$, and Green's formula can be applied, namely, /24

$$\left\langle \frac{1}{c_0^2}\frac{\partial^2\phi}{\partial t^2} - \Delta\phi, \psi \right\rangle = \lim_{\varepsilon \to 0} \int_{-\infty}^{\infty} dt \int_{\Sigma_\varepsilon} \left(\phi \frac{d\psi}{dn} - \psi \frac{d\phi}{dn} \right) dS \qquad (52)$$

where Σ_ε is the interior boundary (sphere or cylinder) of $\mathfrak{D}_\varepsilon$, with the normal derivative $\frac{d}{dn}$ being evaluated in the direction pointing toward $\mathfrak{D}_\varepsilon$. It is immediately obvious that, if a stationary wave is involved, the second term of eq.(52) is zero while it will differ from zero and have the same value if a radiant and an antiradiant wave are present. Thus,

$$\left\langle \frac{1}{c_0^2}\frac{\partial^2\phi}{\partial t^2} - \Delta\phi, \psi \right\rangle = \begin{cases} 0 & \text{for S.W.} \quad a) \\ -\int_{-\infty}^{\infty} \frac{\dot{\mathfrak{M}}(t)}{\rho_0} \psi(r=0, t)\, dt & \text{R.W. or A.W.} \quad b) \end{cases} \qquad (53)$$

which is expressed by writing

$$\frac{1}{c_0^2}\frac{\partial^2\phi}{\partial t^2} - \Delta\phi = \begin{cases} 0 & \text{S.W.} \\ -\dfrac{\mathfrak{M}(t)}{\rho_0}\,\delta_{r=0} & \text{R.W. or A.W.} \end{cases} \tag{54}$$

denoting by $\delta_{r=0}$ the unit Dirac mass at r = 0. Let us note that, no matter what the value of α might be, the function $\alpha\phi_R + (1-\alpha)\,\phi_{AR}$ proves eq.(54b).

1.3 Basic Solution of Cauchy's Problem

1.3.1 Cauchy's Problem

Let us assume that $c_0 = 1$ and write

$$(t, \vec{x}) = (x_0, x_1, x_2, x_3) = (x_0, \vec{x}) \tag{1}$$

so that the problem to be solved will be

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$$\begin{cases} \dfrac{\partial^2\phi}{\partial x_0^2} - \dfrac{\partial^2\phi}{\partial x_1^2} - \dfrac{\partial^2\phi}{\partial x_2^2} - \dfrac{\partial^2\phi}{\partial x_3^2} = 0 \quad , & a) \\ \phi(0,\vec{x}) = F_0(\vec{x}) \quad , & b) \\ \dfrac{\partial\phi(0,\vec{x})}{\partial x_0} = F_1(\vec{x}) \quad , & c) \end{cases} \tag{2}$$

where F_0 and F_1 are as regular as necessary and have a compact support. We then pose

$$\begin{cases} \tilde{\phi} = \begin{cases} \phi & \text{if } x_0 \geqslant 0 \\ 0 & \text{if } x_0 < 0 \end{cases} \\ \mathcal{L} \equiv \dfrac{\partial^2}{\partial x_0^2} - \dfrac{\partial^2}{\partial x_1^2} - \dfrac{\partial^2}{\partial x_2^2} - \dfrac{\partial^2}{\partial x_3^2} \end{cases} \tag{3}$$

yielding

$$\langle \mathcal{L}\tilde{\phi}, \psi\rangle = \langle \tilde{\phi}, \mathcal{L}\psi\rangle = \int_0^\infty dt \iiint_{-\infty}^{\infty} \phi\left(\frac{\partial^2\psi}{\partial x_0^2} - \frac{\partial^2\psi}{\partial x_1^2} - \frac{\partial^2\psi}{\partial x_2^2} - \frac{\partial^2\psi}{\partial x_3^2}\right) d\vec{x} \tag{4}$$

$$= \int_0^\infty dx_0 \iiint \underbrace{\psi\left(\frac{\partial^2\phi}{\partial x_0^2} - \Delta\phi\right)}_{=0} d\mathbf{x} - \iiint F_0 \frac{\partial\psi(0,\mathbf{x})}{\partial x_0} d\mathbf{x} + \iiint F_1 \psi(0,\mathbf{x})\, d\mathbf{x}$$

i.e.:

Theorem 6: If $\Phi(x_0, \mathbf{x})$ is the solution of the Cauchy problem, then $\tilde{\phi}$, which is the extension of ϕ at zero for $x_0 < 0$, proves

$$\mathcal{L}\tilde{\phi} = F_0(\mathbf{x})\,\delta'(x_0) + F_1(\mathbf{x})\,\delta(x_0). \tag{5}$$

Conversely, if $\tilde{\phi}$ proves eq.(5), if it is equal to a function which is twice continuously differentiable in $x_0 \geq 0$, if it vanishes in $x_0 < 0$, if it is continuously differentiable in $x_0 \geq 0$, then $\phi = \tilde{\phi}$ in $x_0 \geq 0$ will be the solution of Cauchy's problem.

The inverse is proved by writing eq.(5) explicitly, in the form /26

$$\int_0^\infty dx_0 \iiint_{-\infty}^{\infty} \psi\left(\frac{\partial^2\phi}{\partial x_0^2} - \Delta\phi\right) d\mathbf{x} - \iiint_{-\infty}^{\infty} \phi(0,\mathbf{x}) \frac{\partial\psi(0,\mathbf{x})}{\partial x_0} d\mathbf{x}$$
$$+ \iiint_{-\infty}^{\infty} \frac{\partial\phi(0,\mathbf{x})}{\partial x_0} \psi(0,\mathbf{x})\, d\mathbf{x} = -\iiint_{-\infty}^{\infty} F_0(\mathbf{x}) \frac{\partial\psi(0,\mathbf{x})}{\partial x_0} d\mathbf{x} \tag{6}$$
$$+ \iiint_{-\infty}^{\infty} F_1(\mathbf{x})\, \psi(0,\mathbf{x})\, d\mathbf{x};$$

after which $\psi(0, \mathbf{x}) = \dfrac{\partial\psi(0, \mathbf{x})}{\partial x_0} = 0$ is selected with $\psi(x_0, \mathbf{x})$ being arbitrary; from this, we deduct $\dfrac{\partial^2\varphi}{\partial x_0^2} - \Delta\phi = 0$, so that the following remains:

$$\iiint_{-\infty}^{\infty} \left\{ \left(F_0 - \phi(0,\mathbf{x})\right) \frac{\partial\psi(0,\mathbf{x})}{\partial x_0} + \left(\frac{\partial\phi(0,\mathbf{x})}{\partial x_0} - F_1\right) \psi(0,\mathbf{x}) \right\} d\mathbf{x} = 0. \tag{7}$$

Since $\dfrac{\partial\psi(0, \mathbf{x})}{\partial x_0}$ as well as $\psi(0, \mathbf{x})$ are arbitrary, it is obvious that eqs.(2a)

and (2b) have taken place.

Let us use the Laplace transformation, by noting

$$\left\{\begin{array}{l} \hat{\phi}(\zeta_\alpha) = \displaystyle\iiiint_{-\infty}^{\infty} e^{-\zeta_\alpha x_\alpha} \phi(x_\alpha)\, dx, \\ \zeta_\alpha = \xi_\alpha + i\eta_\alpha, \qquad \alpha = 0, 1, 2, 3 \\ \zeta_\alpha x_\alpha = \displaystyle\sum_{\alpha=0}^{3} \zeta_\alpha x_\alpha . \end{array}\right. \tag{8}$$

Then, eq.(5) becomes

$$(\zeta_0^2 - \zeta_1^2 - \zeta_2^2 - \zeta_3^2)\,\hat{\phi} = \zeta_0 \hat{F}_0 + \hat{F}_1, \tag{9}$$

where $\hat{F}_0$ and $\hat{F}_1$ do not depend on ζ_0, for example,

$$\hat{F}_0(\zeta_1, \zeta_2, \zeta_3) = \iiint_{-\infty}^{\infty} e^{-(\zeta_1 x_1 + \zeta_2 x_2 + \zeta_3 x_3)} F_0(x_1, x_2, x_3)\, dx_1 dx_2 dx_3 \tag{10}$$

and, since F_0 is regular and vanishes outside of a sphere while $\hat{F}_0$ is analytic for any value of ζ, this represents an integral function. Assuming that F is N times continuously differentiable and will vanish outside of a sphere of radius R, the following incremental series can be constructed: /27

$$|\hat{F}| < \text{Const} \left(1 + |\zeta_0|^2 + |\zeta_2|^2 + |\zeta_3|^2\right)^{-N/2} \exp\left\{R\left(|\xi_1|^2 + |\xi_2|^2 + |\xi_3|^2\right)\right\}. \tag{11}$$

The function

$$\hat{\phi} = \frac{\zeta_0 \hat{F}_0 + \hat{F}_1}{\zeta_0^2 - \zeta_1^2 - \zeta_2^2 - \zeta_3^2} \tag{12}$$

is analytical in any domain that does not contain a denominator root. If

$$\phi(x_0, x_1, x_2, x_3) = \frac{1}{(2\pi)^4} \iiiint_{-\infty}^{\infty} e^{\zeta_0 x_0 + \zeta_1 x_1 + \zeta_2 x_2 + \zeta_3 x_3} \frac{\zeta_0 \hat{F}_0 + \hat{F}_1}{\zeta_0^2 - \zeta_1^2 - \zeta_2^2 - \zeta_3^2}\, d\eta_0 d\eta_1 \ldots d\eta_3 \tag{13}$$

exists as an integral that converges for fixed $\xi_0, \xi_1, \dots \xi_3$, then it will define a function for which eq.(12) is the Laplace transform. Convergence is achieved if the denominator does not vanish, which takes place if

$$\xi_0^2 > \xi_1^2 + \xi_2^2 + \xi_3^2 , \tag{14}$$

since, for $\eta_0, \eta_1, \eta_2, \eta_3 \to \pm\infty$, the convergence is manifest. Let us study this in more detail; posing

$$\xi_1^2 + \xi_2^2 + \xi_3^2 = \xi^2 , \qquad \eta_1^2 + \eta_2^2 + \eta_3^2 = \eta^2 , \qquad \xi_1\eta_1 + \xi_2\eta_2 + \xi_3\eta_3 = \xi\eta\cos\theta$$

we obtain

$$\zeta_0^2 - (\zeta_1^2 + \zeta_2^2 + \zeta_3^2) = \xi_0^2 - \xi^2 + \eta^2 - \eta_0^2 + 2i\left(\xi_0\eta_0 - \xi\eta\cos\theta\right)$$

so that the denominator of eq.(13) can vanish only if

$$\xi_0^2 - \xi^2 = \eta_0^2 - \eta^2 , \qquad \xi_0\eta_0 = \xi\eta\cos\theta ,$$

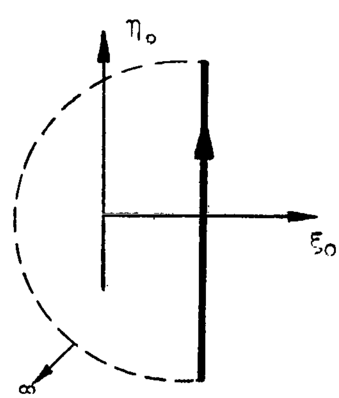
which cannot take place simultaneously under eq.(14); in addition, it is sufficient to study the behavior of

$$\left| \zeta_0^2 - (\zeta_1^2 + \zeta_2^2 + \zeta_3^2) \right|^2 = \left(\eta_0^2 - \eta^2 + \xi^2 - \xi_0^2\right)^2 + 4\left(\xi_0\eta_0 - \xi\eta\cos\theta\right)^2$$

in the case in which $|\eta_0| \to \infty$; by writing $\eta^2 = \eta_0^2 t^2$, we obtain /28

$$\left| \zeta_0^2 - \zeta_1^2 - \zeta_2^2 - \zeta_3^2 \right|^2 = \eta_0^4 \left\{ \left(1 - t^2 + \frac{\xi^2 - \xi_0^2}{\eta_0^2}\right)^2 + \frac{4}{\eta_0^2}\left(\xi_0 - t\xi\cos\theta\right)^2 \right\}$$

and, if $|\eta_0| \to \infty$, the minimum of $\{\ \}$ will take place in the vicinity of $t = 1$, in such a manner that the first term is at least $O(\eta_0^2)$, with the convergence



of eq.(13) being entirely ensured. Let us start the integration by that in η_0, naturally assuming that eq.(14) has taken place. We will perform this calculation with the aid of the theorem of residuals, using the contour of the accompanying diagram; however, a certain choice must be made: $\hat{\phi}$ is analytical either for $\xi_0 > (\xi_1^2 + \xi_2^2 + \xi_3^2)^{\frac{1}{2}}$ or for $\xi_0 < -(\ )^{\frac{1}{2}}$, and the first possibility must be selected since $\hat{\phi}$ is a Laplace transform of ϕ which is $\equiv 0$ for $x_0 < 0$. Then, for $x_0 > 0$, the contribution of the half-circle vanishes when it tends toward infinite, so that the sought value of the integral will be equal to the sum of residues in

$$\zeta_0^* = \pm \sqrt{\zeta_1^2 + \zeta_2^2 + \zeta_3^2}\,. \tag{15}$$

If x_0 were < 0, a zero contribution with the symmetric half-circle would be obtained, indicating that $\phi \equiv 0$ for $x_0 > 0$, as it should be. On evaluating the residues, we find

$$\phi = \frac{1}{2(2\pi)^3} \iiint_{-\infty}^{\infty} \Sigma^* \, e^{x_0 \zeta_0^* + x_i \zeta_i} \left(\frac{\hat{F}_1}{\zeta_0^*} + \hat{F}_0 \right) d\eta_1 \, d\eta_2 \, d\eta_3 \tag{16}$$

where Σ^* indicates that summation must be made over the two roots [eq.(15)]. Let us write the second term of eq.(16) explicitly, by posing

$$\boldsymbol{\eta} = (\eta_1, \eta_2, \eta_3) = \eta\,\omega, \quad \omega = \text{unit vector}$$

and by selecting $\xi_1 = \xi_2 = \xi_3 = 0$, which can be done since the result is independent of the selection as long as eq.(14) has taken place, as results from Cauchy's theorem; we then obtain /29

$$\phi = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} e^{i\eta\, \boldsymbol{x}\cdot\omega} \left(\hat{F}_0(\eta\omega) \cos(x_0\eta) + \hat{F}_1(\eta\omega) \frac{\sin(x_0\eta)}{\eta} \right) d\boldsymbol{\eta}. \tag{17}$$

Let us now denote by T_{x_0} the distribution of the vectorial variable $\boldsymbol{x}$ whose Fourier transform is equal to $\frac{\sin(\eta x_0)}{\eta}$, so that $\frac{\partial T_{x_0}}{\partial x_0}$ admits, as the Fourier transform, $\cos(\eta x_0)$ and eq.(17) means that

$$\phi = F_0 * \frac{\partial T_{x_0}}{\partial x_0} + F_1 * T_{x_0} \tag{18}$$

denoting, by an asterisk * the convolution product on the Cartesian coordinates

(x_1, x_2, x_3). It is now necessary to calculate T_{x_0}, but there exists a difficulty since this calculation cannot be performed by means of eq.(17) with $F_0 = 0$, $F_1 = 1$, thus necessitating a round-about procedure. Let ψ be the basic function on which T_{x_0} is made to act and let $\hat{\psi}$ be the Fourier transform of ψ (here, we have functions of $\boldsymbol{x}$), which will yield

$$\iiint \hat{T}_{x_0}(\eta)\hat{\psi}^*(\eta)\, d\eta = (2\pi)^3 \langle T_{x_0}, \psi \rangle \tag{19}$$

using an asterisk * for denoting the operation of passing to the complex conjugate. The definition formula [eq.(16)] can be explicitly written as follows:

$$\begin{aligned}\langle T_{x_0}, \psi \rangle &= \frac{1}{(2\pi)^3}\int_0^\infty \eta \sin(x_0\eta)\, d\eta \iint \hat{\psi}^*(\eta\omega)\, d\omega \\ &= \frac{1}{(2\pi)^3}\int_0^\infty \eta \sin(x_0\eta)\, d\eta \iint d\omega \int_0^\infty r^2 dr \iint d\Theta\; e^{-i\eta r\,\omega\Theta}\, \psi(r\Theta)\end{aligned} \tag{20}$$

where Θ is a unit vector such that $\boldsymbol{x} = r\,\Theta$. It should be mentioned that, in view of the fact that ψ vanishes outside of a sphere, the integral in r actually is an integral between finite limits, so that the three last integrations can be interchanged at will; this yields /30

$$\langle T_{x_0}, \psi \rangle = \frac{2}{(2\pi)^2}\int_0^\infty \sin(\eta x_0)\, d\eta \int_0^\infty r \sin(\eta r)\, dr \iint \psi(r\Theta)\, d\Theta. \tag{21}$$

Let us note:

$\mathfrak{M}_r^{\boldsymbol{x}}(\psi)$ = mean of ψ on the sphere centered in $\boldsymbol{x}$ with a radius r; (21)

from this, we obtain

$$\langle T_{x_0}, \psi \rangle = \frac{2}{\pi}\int_0^\infty \sin(\eta x_0)\, d\eta \int_0^\infty r \sin(\eta r)\, F(r)\, dr$$

$$F(r) = \mathfrak{M}_r^0(\psi)$$

noting that F(r) vanishes outside of a finite interval; let us also note that

F(r) is very regular so that the usual conditions prevail for applying the inverse formula of the Fourier integral (real with sin), yielding

$$< T_{x_0}, \psi > = x_0 \mathfrak{M}_{x_0}^{0}(\psi). \tag{22}$$

Theorem 7: The distribution S_{x_0} which, at any function ψ, associates its mean value on the sphere with the center of origin and the radius x_0, has the following function of $\eta = (\eta_1^2 + \eta_2^2 + \eta_3^2)^{\frac{1}{2}}$ as Fourier transform:

$$\mathcal{F}(S_{x_0}) = \frac{\sin(\eta x_0)}{\eta x_0}. \tag{23}$$

Theorem 8: The solution of Cauchy's problem [eq.(2)] is given by the following formula, derived by Poisson:

$$\phi(x_0, x) = x_0 \mathfrak{M}_{x_0}^{x}(F_1) + \frac{\partial}{\partial x_0} x_0 \mathfrak{M}_{x_0}^{x}(F_0), \tag{24}$$

where $\mathfrak{M}_{x_0}^{x}(f)$ denotes the operation consisting in using the mean value of f on the sphere with a center x and a radius x_0. /31

The proof is as follows: According to eq.(18), we have

$$< \phi, \psi > = I_1(x_0) + \frac{\partial I_0(x_0)}{\partial x_0}$$

with

$$I_{0,1}(x_0) = \iiint dx \iint d\omega \quad F_{0,1}(x)\, \psi(x + x_0\omega)\, x_0$$

$$= \iiint dx \iint d\omega \quad F_{0,1}(x + x_0\omega)\, \psi(x)\, x_0$$

because of an evident change in variable. We leave to the reader to make this change, which requires no calculation.

1.3.2 Proof of Poisson's Formula; Two-Dimensional Passage

Let us pose

$$\begin{cases} \phi_1(x_0, x) = x_0 \mathfrak{M}_{x_0}^{x}(F_1), \\ \phi_0(x_0, x) = \dfrac{\partial}{\partial x_0}\left\{x_0 \mathfrak{M}_{x_0}^{x}(F_0)\right\}, \end{cases} \tag{25}$$

and let us start by studying the question of initial conditions. Obviously, if f is continuously differentiable, we will have, for example,

$$\left\{\begin{aligned} &\lim_{x_0 \to 0} \mathcal{M}_{x_0}^{\mathbf{x}}(f) = f(\mathbf{x}), \\ &\lim_{x_0 \to 0} \frac{\partial}{\partial x_0} \mathcal{M}_{x_0}^{\mathbf{x}}(f) = 0, \end{aligned}\right. \tag{26}$$

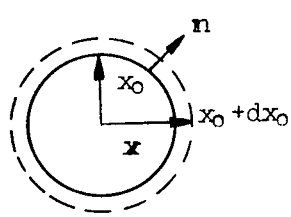
and, if f is twice continuously differentiable,

$$\lim_{x_0 \to 0} \frac{\partial^2 \mathcal{M}_{x_0}^{\mathbf{x}}(f)}{\partial x_0^2} = \frac{1}{3} \Delta f, \tag{27}$$

as can be done by an expansion in a Taylor series, in the vicinity of $\mathbf{x}$. This demonstrates that we have /32

$$\left\{\begin{aligned} &\phi_0(0,\mathbf{x}) = F_0(\mathbf{x}), \\ &\frac{\partial \phi_0(0,\mathbf{x})}{\partial x_0} = 0, \end{aligned}\right. \qquad \left\{\begin{aligned} &\phi_1(0,\mathbf{x}) = 0, \\ &\frac{\partial \phi_1(0,\mathbf{x})}{\partial x_0} = F_1(\mathbf{x}), \end{aligned}\right. \tag{28}$$

so that the Poisson solution proves the initial conditions if F_0 is twice continuously differentiable and if F_1 is once continuously differentiable. Passing



from this to a proof of the equation, we have

$$\frac{\partial}{\partial x_0}\left\{\mathcal{M}_{x_0}^{\mathbf{x}}(f)\right\} = \mathcal{M}_{x_0}^{\mathbf{x}}(\mathbf{n} \cdot \nabla f), \tag{29}$$

as is directly obvious from looking at the accompanying diagram. For the second derivative, it is sufficient to repeat the process, i.e.,

$$\frac{\partial^2}{\partial x_0^2}\left\{\mathcal{M}_{x_0}^{\mathbf{x}}(f)\right\} = \mathcal{M}_{x_0}^{\mathbf{x}}\left(\mathbf{n} \cdot \nabla [\mathbf{n} \cdot \nabla f]\right) = \mathcal{M}_{x_0}^{\mathbf{x}}\left(\mathbf{n}\,\mathbf{n} : \nabla\nabla f\right) \tag{30}$$

with $\mathbf{n}\,\mathbf{n}: \nabla\nabla f = n_i n_j \dfrac{\partial^2 f}{\partial x_i \partial x_j}$ and summation. Then,

$$\Delta\left(\mathcal{M}_{x_0}^{\mathbf{x}}(f)\right) = \Delta\left\{\frac{1}{4\pi}\iint f(\mathbf{x}+x_0\omega)\,d\omega\right\} = \frac{1}{4\pi}\iint \Delta f(\mathbf{x}+x_0\omega)\,d\omega \tag{31}$$
$$= \mathcal{M}_{x_0}^{\mathbf{x}}(\Delta f),$$

and, more accurately, any differentiation with respect to a coordinate x_1, x_2, x_3 commutates with the operation of the mean. Using the above results, we find

$$\left(\frac{\partial^2}{\partial x_0^2}-\Delta\right)\left\{x_0\,\mathcal{M}_{x_0}^{\mathbf{x}}(f)\right\} = 2\,\mathcal{M}_{x_0}^{\mathbf{x}}(\mathbf{n}\cdot\nabla f) + \tag{32}$$
$$+\, x_0\,\mathcal{M}_{x_0}^{\mathbf{x}}\left(\mathbf{n}\,\mathbf{n}:\nabla\nabla f - \Delta f\right)$$

while, by applying the theorem of divergence, we find

$$x_0^2\,\mathcal{M}_{x_0}^{\mathbf{x}}(\mathbf{n}\cdot\nabla f) = \int_0^{x_0}\lambda^2\,\mathcal{M}_{\lambda}^{\mathbf{x}}(\Delta f)\,d\lambda \tag{33}$$

so that it becomes obvious that

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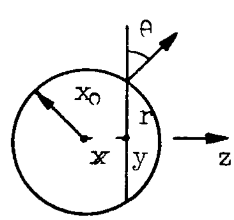
$$\left(\frac{\partial^2}{\partial x_0^2}-\Delta\right)\left\{x_0\,\mathcal{M}_{x_0}^{\mathbf{x}}(f)\right\} = 0. \tag{34}$$

Thus, we have verified that each of the functions $\varphi_0\,(x_0, \mathbf{x})$ and $\varphi_1\,(x_0, \mathbf{x})$, defined by eq.(25), is a solution of the wave equation if F_0 and F_1 are twice continuously differentiable.

We will now apply the method of steepest descent to solving the Cauchy problem, if only two-dimensional space is present, applying Poisson's formula



to the case in which F_0 and F_1 are constant along straight lines. We will select the coordinate axes, with oz parallel to the direction in question,

noting that

$$\overset{(3)x}{\mathfrak{M}}_{x_0}(f) = \frac{1}{2}\int_{-x_0}^{x_0} dz \; \overset{(2)y}{\mathfrak{M}}_{x_0\cos\theta}\left(\frac{f}{\cos\theta}\right)\frac{z\cos\theta}{x_0^2} \qquad y = x + zK \tag{35}$$

for a function f which does not depend on z, using the notations of the accompanying diagram for θ and a superscript in parentheses for indicating the number of dimensions of space. Still using the notations of the diagram, we have $\sin\theta = \sqrt{1 - \frac{r^2}{x_0^2}}$, so that

$$\overset{(3)x}{\mathfrak{M}}_{x_0}(f) = \frac{1}{2\pi x_0}\iint \frac{f(x',y')}{\sqrt{x_0^2-(x-x')^2-(y-y')^2}}\,dx'dy' \tag{36}$$

where the integral extends over the disk $(x - x')^2 + (y - y')^2 < x_0^2$.

Theorem 9: The solution of Cauchy's problem, in accordance with

$$\begin{cases} \dfrac{\partial^2\phi}{\partial x_0^2} - \dfrac{\partial^2\phi}{\partial x^2} - \dfrac{\partial^2\phi}{\partial y^2} = 0 \\ \phi(0,x,y) = F_0(x,y), \\ \dfrac{\partial\phi(0,x,y)}{\partial x_0} = F_1(x,y), \end{cases} \tag{37}$$

is given by the following formula:

$$\phi(x_0,x,y) = \frac{1}{2\pi}\iint_{\Delta(x_0,x)} \frac{F_1(x',y')\,dx'dy'}{\sqrt{x_0^2-(x-x')^2-(y-y')^2}} + \frac{1}{2\pi}\frac{\partial}{\partial x_0}\iint_{\Delta(x_0,x)} \frac{F_0(x',y')\,dx'dy'}{\sqrt{x_0^2-(x-x')^2-(y-y')^2}} \tag{38}$$

where $\Delta(x_0, x, y)$ is the disk $(x - x')^2 + (y - y')^2 < x_0^2$. /34

Let us now select

$$F_0 = \frac{\alpha_0}{\pi\varepsilon}\,e^{-\frac{x^2+y^2}{\varepsilon}}, \qquad F_1 = \frac{\alpha_1}{\pi\varepsilon}\,e^{-\frac{x^2+y^2}{\varepsilon}},$$

where α_0, α_1, ϵ are constants and let us pass to the limit $\epsilon \to 0$, thus obtaining:

Theorem 10: The distribution T_{α_0,α_1}, depending on the parameters α_0 and α_1 and defined by

$$\langle T_{\alpha_0,\alpha_1}, \varphi \rangle = \frac{\alpha_1}{2\pi}\int_0^\infty dx_0 \iint_{\Delta(x_0,0)} \frac{\varphi(x_0,x,y)}{\sqrt{x_0^2-x^2-y^2}}\,dx\,dy - \frac{\alpha_1}{2\pi}\int_0^\infty dx_0 \iint_{\Delta(x_0,0)} \frac{\frac{\partial}{\partial x_0}\varphi(x_0,x,y)}{\sqrt{x_0^2-x^2-y^2}}\,dx\,dy \tag{39}$$

is the solution of

$$\frac{\partial^2 T_{\alpha_0,\alpha_1}}{\partial x_0^2} - \frac{\partial^2 T_{\alpha_0,\alpha_1}}{\partial x^2} - \frac{\partial^2 T_{\alpha_0,\alpha_1}}{\partial y^2} = \alpha_0\,\delta(x_0)\delta(x)\delta(y) + \alpha_1\,\delta'(x_0)\delta(x)\delta(y). \tag{40}$$

Let $\mathfrak{M}(t)$ be a function, so that the distribution

$$T_{\mathfrak{M}} = -\mathfrak{M}(x_0) * T_{0,1}(x_0,x,y), \tag{41}$$

where the convolution refers to the variable x_0, will prove

$$\frac{\partial^2 T_{\mathfrak{M}}}{\partial x_0^2} - \frac{\partial^2 T_{\mathfrak{M}}}{\partial x^2} - \frac{\partial^2 T_{\mathfrak{M}}}{\partial y^2} = -\mathfrak{M}(x_0)\,\delta(x)\,\delta(y). \tag{42}$$

Let us write $T_{\mathfrak{M}}$ explicitly, yielding

$$\langle T_{\mathfrak{M}}, \varphi\rangle = -\frac{1}{2\pi}\int_0^\infty dx_0 \int_{-\infty}^\infty d\xi_0\, \mathfrak{M}(\xi_0) \iint_{\Delta(x_0,0)} \frac{\varphi(x_0+\xi_0,x,y)}{\sqrt{x_0^2-x^2-y^2}}\,dx\,dy = -\frac{1}{2\pi}\int_{-\infty}^\infty dx_0 \int_{-\infty}^{x_0} d\xi_0 \iint_{\Delta(x_0-\xi_0,0)} \frac{\mathfrak{M}(\xi_0)\,\varphi(x_0,x,y)}{\sqrt{(x_0-\xi_0)^2-x^2-y^2}}\,dx\,dy, \tag{43}$$

so that, if $\mathfrak{M}$ is a continuous function, we will have /35

$$T_{\mathfrak{M}_0} = -\frac{1}{2\pi}\int_{-\infty}^{x_0} \frac{\mathfrak{M}(\xi_0)}{\sqrt{(x_0-\xi_0)^2 - x^2 - y^2}}\, dx\, dy. \tag{44}$$

Problem 8: 1) Apply Poisson's equation to the case in which F_0 and F_1 depend neither on y nor on z. Find the same result by starting this time from eq.(38) and by assuming that F_0 and F_1 do not depend on y.

2) Demonstrate that, if ϕ is a solution of the wave equation, $r\mathfrak{M}_r^{\boldsymbol{x}}\{\phi(x_0, \boldsymbol{x})\} = F(x_0, r)$ does not depend on $\boldsymbol{x}$, and then prove the equation $\frac{\partial^2 F}{\partial x_0^2} - \frac{\partial^2 F}{\partial r^2} = 0$.

3) Assuming that $F_0(\boldsymbol{x}) = \frac{F(r)}{r}$ for $r_1 \leq r \leq r_2$ and that it vanishes anywhere else as soon as $F_1(\boldsymbol{x}) \equiv 0$, find ϕ by applying Poisson's equation and find the same result by making use of Section 1.2.2.

1.3.3 Retarded Potentials

Let us now consider a problem related to that considered in Section 1.3.1. Let

$$\begin{cases} \dfrac{\partial^2\phi}{\partial x_0^2} - \Delta\phi = G(x_0, \boldsymbol{x}), & G = \text{zero at infinity} \quad (x_0 > 0) \\ \phi(0, \boldsymbol{x}) = 0, & \\ \dfrac{\partial\phi(0,\boldsymbol{x})}{\partial x_0} = 0. & \end{cases} \tag{45}$$

By passing over the intermediary of the function $\tilde{\phi}$ which extends ϕ through 0 at $x_0 \leq 0$ and by effecting a Laplace transformation, we will obtain

$$(\zeta_0^2 - \zeta_1^2 - \zeta_2^2 - \zeta_3^2)\,\hat{\phi}(\zeta_0, \zeta_1, \zeta_2, \zeta_3) = \hat{G}(\zeta_0, \zeta_1, \zeta_2, \zeta_3), \tag{46}$$

and, as above, /36

$$\begin{aligned} \phi &= \frac{1}{(2\pi)^4}\iiiint_{-\infty}^{\infty} e^{\zeta_0 x_0 + \zeta\cdot\boldsymbol{x}}\,\frac{\hat{G}(\zeta_0, \zeta)}{\zeta_0^2 - \zeta^2}\, d\eta_0\, d\eta \\ &= \frac{1}{(2\pi)^3\cdot 2}\iiint_{-\infty}^{\infty} \sum{}^{*} e^{x_0\zeta_0^* + \zeta\cdot\boldsymbol{x}}\,\frac{\hat{G}(\zeta_0^*, \zeta)}{\zeta_0^*}\, d\eta \end{aligned} \tag{47}$$

$$= \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} e^{i\eta\, x\cdot\omega}\, \hat{G}(\eta, \eta\omega) \frac{\sin(\eta x_0)}{\eta}\, d\eta \,.$$

Let us pose

$$H(x) = \frac{1}{(2\pi)^3} \iiint_{-\infty}^{\infty} e^{i\eta\, x\cdot\omega}\, \hat{G}(\eta, \eta\omega)\, d\eta \,, \tag{48}$$

so that we obtain

$$\phi(x_0, x) = T_{x_0} * H = x_0\, M^{x}_{x_0}(H) \tag{49}$$

because of theorem 7. Writing this explicitly, we will have

$$\phi = \frac{x_0}{8\pi^2} \iint \frac{d\Theta}{2\pi} \int_0^{\infty} \eta^2 d\eta \iint \frac{d\omega}{2\pi} \iiiint_{-\infty}^{\infty} G(x_0', x') \exp\left\{ i\eta\left[(x - x')\omega + x_0\Theta\cdot\omega - x_0'\right]\right\} dx_0' dx' . \tag{50}$$

We will then calculate the second term by first making the integrations in Θ and ω in sequence. In fact, the quadrupule integral in x_0' and x' is a function of η which vanishes rather rapidly at infinity, thus ensuring convergence of the integral in η. Since we assume that G is regular, it follows from this that the integration in Θ can be permuted with that in η; after this, under the condition that the integration in η is always performed last, it is possible to perform the other integrations in any desired order since these are integrals over a finite domain, in view of the fact that G has a bounded base. We leave to the reader the task of making the integration $\iint d\omega \iint d\Theta$ in the proper order and thus to find that

$$\phi = \frac{1}{2\pi^2} \int_0^{\infty} \sin(\eta x_0)\, d\eta \int_{-\infty}^{\infty} e^{i\eta(y_0 - x_0)} dy_0 \iiint_{-\infty}^{\infty} \frac{\sin(\eta|y|)}{|y|} G(x_0 + y_0, x + y)\, dy . \tag{51}$$

The volume integral is transformed by making use of the operation of the spherical mean, namely, /37

$$\phi = \frac{2}{\pi} \int_0^{\infty} \sin(\eta x_0)\, d\eta \int_{-\infty}^{\infty} e^{i\eta(y_0 - x_0)} dy_0 \int_0^{\infty} r \sin(\eta r)\, M_r^{x,\, x_0 - y_0}(G)\, dr \tag{52}$$

with an obvious notation. We will then write $\int_0^{\infty} \sin(\eta x_0)\, d\eta = \frac{1}{i} \int_{-\infty}^{\infty} e^{i\eta x_0} d\eta$, after which we note that, as a function of η, the last integral vanishes at infinity as rapidly as desired if G is sufficiently regular, which permits interchanging $\int d\eta$ and $\int dy_0$, so to arrive finally at

$$\phi = \frac{2}{\pi}\int_{-\infty}^{\infty} dy_0 \int_0^{\infty} \sin(\eta y_0)\, d\eta \int_0^{\infty} r \sin(\eta r)\, \mathfrak{M}_r^{\boldsymbol{x},\, x_0 - y_0}(G)\, dr, \tag{53}$$

in a form in which the Fourier reciprocity principle is applicable:

$$\phi = \int_0^{\infty} r\, \mathfrak{M}_r^{\boldsymbol{x},\, x_0 - r}(G)\, dr. \tag{54}$$

Theorem 11: Since the conditions of theorem 8 are satisfied and if, in addition, the function $G(x_0, \boldsymbol{x})$ is twice continuously differentiable*, then the function

$$\phi(x_0, \boldsymbol{x}) = x_0\, \mathfrak{M}_{x_0}^{\boldsymbol{x}}(F_1) + \frac{\partial}{\partial x_0}\left\{x_0\, \mathfrak{M}_{x_0}^{\boldsymbol{x}}(F_0)\right\} + \int_0^{\infty} r\, \mathfrak{M}_r^{\boldsymbol{x},\, x_0 - r}(G)\, dr \tag{55}$$

proves the equation

$$\frac{\partial^2\phi}{\partial x_0^2} - \frac{\partial^2\phi}{\partial x_1^2} - \frac{\partial^2\phi}{\partial x_2^2} - \frac{\partial^2\phi}{\partial x_3^2} = G(x_0, \boldsymbol{x}) \tag{56}$$

as well as the conditions (2b) and (2c). The expression

$$4\pi \int_0^{\infty} r\, \mathfrak{M}_r^{\boldsymbol{x},\, x_0 - r}(G)\, dr = \iiint_{-\infty}^{\infty} \frac{G(x_0 - |\boldsymbol{x} - \boldsymbol{x}'|, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|}\, d\boldsymbol{x}' \tag{57}$$

is known as "retarded potential".

Notation: To facilitate writing of the formulas, the following notation is frequently used: /38

---

* vanishing identically at $x_0 < 0$.

$$[f](x_0, \mathbf{x}; \mathbf{x}') = f(x_0 - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}') \tag{58}$$

Thus, the retarded potential of f will be

$$\iiint_{-\infty}^{\infty} [f](x_0, \mathbf{x}; \mathbf{x}')\, d\mathbf{x}'.$$

Verification of the Formula for Retarded Potentials:

It is sufficient to prove that

$$\left(\frac{\partial^2}{\partial x_0^2} - \Delta\right) \iiint_{-\infty}^{\infty} \frac{G(x_0 - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{4\pi |\mathbf{x} - \mathbf{x}'|}\, d\mathbf{x}' = G(x_0, \mathbf{x}) \tag{59}$$

For this, we will demonstrate that

$$\left\langle \frac{1}{4\pi} \iiint_{-\infty}^{\infty} \frac{G(x_0 - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}\, d\mathbf{x}', \frac{\partial^2 \varphi}{\partial x_0^2} - \Delta \varphi \right\rangle = \iiiint_{-\infty}^{\infty} G(x_0, \mathbf{x})\, \varphi(x_0, \mathbf{x})\, dx_0\, d\mathbf{x} \tag{60}$$

and more accurately,

$$\lim_{\varepsilon \to 0} \iiiiiint_{\mathcal{D}_\varepsilon} \frac{G(x_0 - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{4\pi |\mathbf{x} - \mathbf{x}'|} \left(\frac{\partial^2 \varphi(x_0, \mathbf{x})}{\partial x_0^2} - \Delta \varphi(x_0, \mathbf{x})\right) d\mathbf{x}'\, dx_0\, d\mathbf{x} = \iiiint_{-\infty}^{\infty} G(x_0, \mathbf{x})\, \varphi(x_0, \mathbf{x})\, dx_0\, d\mathbf{x}, \tag{61}$$

where the domain $\mathcal{D}_\varepsilon$ is defined by $|\mathbf{x} - \mathbf{x}'| > \epsilon$, without any restriction. In this domain, we have

$$\left(\frac{\partial^2}{\partial x_0^2} - \Delta\right) \frac{G(x_0 - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} = 0 \tag{62}$$

since, for a fixed $\mathbf{x}'$, the quantity $\dfrac{G(x_0 - \mathbf{x} - \mathbf{x}', \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|}$ is a spherical wave, where G is twice continuously differentiable. By applying Green's function, the

first term of eq.(61) is brought to the form of

$$\lim_{\varepsilon\to 0}\frac{1}{4\pi}\iiint_{-\infty}^{\infty} dx' \int_{-\infty}^{\infty} dx_0 \iint_{\Sigma_\varepsilon}\left\{\psi(x_0, x)\frac{d}{dn}\frac{[G](x_0, x; x')}{|x-x'|} - \frac{[G]}{|x-x'|}\frac{d\psi}{dn}\right\} dS' . \tag{63}$$

where Σ_ε is the sphere $|x - x'| = \varepsilon$ and $\frac{d}{dn}$ is the normal exterior derivative.

The passage to the limit is effected immediately, yielding eq.(61) quite accurately. /39

1.3.4 Elementary Solution

The designation "elementary solution" is given to a distribution $\mathfrak{G}_R(x_0, x)$ such that, in the sense of distributions, we will have

$$\frac{\partial^2 \mathcal{E}_R}{\partial x_0^2} - \Delta \mathcal{E}_R = \delta(x_0)\,\delta(x_1)\,\delta(x_2)\,\delta(x_3), \tag{64}$$

so that

$$\left(\frac{\partial^2}{\partial x_0^2} - \Delta\right)\left(\mathcal{E}_R * G\right) = G . \tag{65}$$

The theorem of retarded potentials permits finding $\mathfrak{G}$. In fact, we have

$$\langle \mathcal{E}_R * G, \psi\rangle = \frac{1}{4\pi}\iiiint\!\!\iiint_{-\infty}^{\infty} \frac{G(x_0 - |x-x'|, x')}{|x-x'|}\psi(x_0, x)\, dx_0\, dx\, dx'$$

$$= \frac{1}{4\pi}\iiiint\!\!\iiint_{-\infty}^{\infty} G(x_0, x)\frac{\psi(x_0 + |y|, x+y)}{|y|}\, dx_0\, dx\, dy . \tag{66}$$

$$= \iiiint_{-\infty}^{\infty} dx_0\, dx\, G(x_0, x)\langle \mathcal{E}_R(y_0, y), \psi(x_0 + y_0, x + y)\rangle,$$

which demonstrates that the distribution of $\mathfrak{G}$ is defined by

$$
\begin{aligned}
\langle \xi_R, \psi \rangle &= \frac{1}{4\pi} \iiint_{-\infty}^{\infty} \frac{\psi(|y|, y)}{|y|}\, dy \\
&= \frac{1}{4\pi} \int_0^{\infty} x_0\, dx_0 \iint \psi(x_0, x_0\omega)\, d\omega \ ,
\end{aligned}
\tag{67}
$$

and it is obvious that

$$
\xi_R = 0 \ , \qquad x_0 < 0 \ . \tag{68}
$$

By analogy with the terminology used for spherical and cylindrical waves, it /40 can be stated that $\mathfrak{E}_R$ is the elementary radiant solution. It is also possible to define an elementary antiradiant solution by

$$
\begin{aligned}
\langle \xi_{AR}, \psi \rangle &= \frac{1}{4\pi} \iiint_{-\infty}^{\infty} \frac{\psi(-|y|, y)}{|y|}\, dy \\
&= \frac{1}{4\pi} \int_{-\infty}^{0} x_0\, dx_0 \iint \psi(x_0, -x_0\omega)\, d\omega
\end{aligned}
\tag{69}
$$

where we will have, this time,

$$
\xi_{AR} = 0 \ , \qquad x_0 > 0 \ . \tag{70}
$$

Finally

$$
\xi = \frac{1}{2}\, \xi_R + \frac{1}{2}\, \xi_{AR} \tag{71}
$$

is still another elementary solution, characterized by

$$
\langle \xi, \psi \rangle = \frac{1}{8\pi} \iiint_{-\infty}^{\infty} \frac{\psi(|y|, y) + \psi(-|y|, y)}{|y|}\, dy \ . \tag{72}
$$

Using $\mathfrak{G}$, we will obtain, with

$$\mathcal{E} * G = \frac{1}{8\pi}\iiint_{-\infty}^{\infty} \frac{G(x_0+|\mathbf{x}-\mathbf{x}'|,\mathbf{x}')+G(x_0-|\mathbf{x}-\mathbf{x}'|,\mathbf{x}')}{|\mathbf{x}-\mathbf{x}'|}\,d\mathbf{x}' \tag{73}$$

a solution for the wave equation which combines the advanced and retarded potentials.

We will next modify the expression of $\mathfrak{G}_R$, by introducing the function

$$\Gamma = x_0^2 - x_1^2 - x_2^2 - x_3^2 \tag{74}$$

and considering the distribution $\frac{\delta(\Gamma)}{4\pi}$. According to the definition formula /41 itself, we have

$$\left\langle \frac{\delta(\Gamma)}{4\pi}, \varphi \right\rangle = \frac{1}{8\pi}\iiint_{-\infty}^{\infty}\left(\frac{\varphi(|\mathbf{x}|,\mathbf{x})}{|\mathbf{x}|}+\frac{\varphi(-|\mathbf{x}|,\mathbf{x})}{\mathbf{x}}\right)d\mathbf{x} \tag{75}$$

since we can select $\mathbf{x}$ and Γ, as new variables instead of $\mathbf{x}$ and x_0 and since the Jacobian of the change in variables is $\left|\frac{\partial\Gamma}{\partial x_0}\right|^{-1}$ which has a value of $1/2|\mathbf{x}|$ for $\Gamma = 0$.

By using the function

$$1(x_0) = \begin{cases} 1 & \text{if } x_0 > 0 \\ 0 & \text{if } x_0 < 0 \end{cases} \tag{76}$$

we obtain the following theorem:

<u>Theorem 12</u>: The elementary solutions of the wave equation of three different types, namely, radiant, antiradiant, and combined, are given by

$$\begin{cases} \mathcal{E}_R = \dfrac{\delta(\Gamma)}{2\pi}\, 1(x_0), & a) \\ \mathcal{E}_{AR} = \dfrac{\delta(\Gamma)}{2\pi}\, 1(-x_0), & b) \end{cases} \tag{77}$$

$$\Big(\quad \xi = \frac{\delta(\Gamma)}{4\pi}, \qquad\qquad \text{c)}$$

each of which proves

$$\frac{\partial^2 \xi}{\partial x_0^2} - \frac{\partial^2 \xi}{\partial x_1^2} - \frac{\partial^2 \xi}{\partial x_2^2} - \frac{\partial^2 \xi}{\partial x_3^2} = \delta(x_0)\,\delta(x_1)\,\delta(x_2)\,\delta(x_3)\,. \tag{78}$$

Let us consider the distribution $\frac{\delta(\Gamma)}{2\pi}$ and let us apply the wave operator to this distribution, thus yielding

$$\left(\frac{\partial^2}{\partial x_0^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_3^2}\right)\frac{\delta(\Gamma)}{4\pi} = \frac{1}{\pi}\left\{\Gamma\,\delta''(\Gamma) + 2\,\delta'(\Gamma)\right\}. \tag{79}$$

The second term of this equation is a distribution concentrated in $x_0 = x_1 = x_2 = x_3 = 0$ by virtue of the relations

$$< F\,\delta^{(k)}(F) + k\,\delta^{(k-1)}(F),\ \psi > \ = 0 \tag{80}$$

for any function ψ which is zero in the vicinity of the singular points of the /42
surface $F = 0$. We will next demonstrate that the second term of eq.(79) is actually equal to $\delta(x_0)\delta(x_1)\delta(x_2)\delta(x_3)$. Let ψ be a function which is identically zero in the vicinity of x_0 and let us pose $\bar{\psi}(\Gamma, \boldsymbol{x}) = \psi(\sqrt{\Gamma + |\boldsymbol{x}|^2}, \boldsymbol{x}) + \psi(-\sqrt{\Gamma + |\boldsymbol{x}|^2}, \boldsymbol{x})$, which will yield

$$\begin{aligned} < \frac{1}{\pi}\left(\Gamma\,\delta''(\Gamma) + 2\,\delta'(\Gamma)\right), \psi > &= < \frac{\delta(\Gamma)}{4\pi}, \frac{\partial^2\psi}{\partial x_0^2} - \frac{\partial^2\psi}{\partial x_1^2} - \frac{\partial^2\psi}{\partial x_2^2} - \frac{\partial^2\psi}{\partial x_3^2} > = \\ &= \frac{1}{\pi}\iiint_{-\infty}^{\infty} d\boldsymbol{x}\left\{\frac{\partial^2}{\partial\Gamma^2}\left[\frac{\bar{\psi}(\Gamma,\boldsymbol{x})\,\Gamma}{\sqrt{|\boldsymbol{x}|^2+\Gamma}}\right] - 2\,\frac{\partial}{\partial\Gamma}\left[\frac{\bar{\psi}(\Gamma,\boldsymbol{x})}{\sqrt{|\boldsymbol{x}|^2+\Gamma}}\right]\right\}_{\Gamma=0} \\ &= \frac{3}{4\pi}\iiint_{-\infty}^{\infty} d\boldsymbol{x}\left\{\frac{\bar{\psi}(\Gamma,\boldsymbol{x})}{\left(|\boldsymbol{x}|^2+\Gamma\right)^{5/2}}\right\}_{\Gamma=0} \end{aligned} \tag{81}$$

Although the calculation, in its present form, makes use of the fact that ψ vanishes in the vicinity of $x_0 = x_1 = x_2 = x_3 = 0$, the relation

$$\left\langle \frac{\delta(\Gamma)}{4\pi}, \frac{\partial^2\psi}{\partial x_0^2} - \frac{\partial^2\psi}{\partial x_1^2} - \frac{\partial^2\psi}{\partial x_2^2} - \frac{\partial^2\psi}{\partial x_3^2} \right\rangle = \frac{3}{4\pi} \iiint_{-\infty}^{\infty} d\mathbf{x} \left\{ \frac{\bar{\psi}(\Gamma, \mathbf{x})\Gamma}{(|\mathbf{x}|^2 + \Gamma)^{5/2}} \right\}_{\Gamma=0} \tag{82}$$

is generally valid since the two members of this equation can be defined by passage to the limit. Thus, let us assume that ψ does not vanish in $x_0 = x_1 = x_2 = x_3 = 0$, so that we have

$$\begin{aligned} \lim_{\Gamma \to 0} \iiint_{-\infty}^{\infty} d\mathbf{x} \frac{\bar{\psi}(\Gamma, \mathbf{x})\Gamma}{(|\mathbf{x}|^2 + \Gamma)^{5/2}} &= \lim_{\Gamma \to 0} \iiint_{-\infty}^{\infty} d\mathbf{x} \frac{\bar{\psi}(\Gamma, \Gamma\mathbf{x})}{(1 + |\mathbf{x}|^2)^{5/2}} \\ &= 4\pi\psi(0, \mathbf{0}) \int_0^{\infty} \frac{u^2}{(u^2+1)^{5/2}} du = \frac{4\pi}{3} \psi(0, \mathbf{0}) \end{aligned} \tag{83}$$

which achieves verification of eq.(79). We leave to the reader the task of proving eqs.(77a) and (77b).

Problem 9: Demonstrate that we have /43

$$\left\{ \begin{aligned} \mathcal{E}_R * \mathcal{M}(x_0) &= \frac{1}{2\pi} \int_{-\infty}^{x_0 - |\mathbf{x}|} \frac{\mathcal{M}(\xi_0)}{\sqrt{(x_0 - \xi_0)^2 - |\mathbf{x}|^2}} d\xi_0 , \\ \mathcal{E}_{AR} * \mathcal{M}(x_0) &= \frac{1}{2\pi} \int_{x_0 + |\mathbf{x}|}^{\infty} \frac{\mathcal{M}(\xi_0)}{\sqrt{(\xi_0 - x_0)^2 - |\mathbf{x}|^2}} d\xi_0 , \end{aligned} \right. \tag{84}$$

where the operation of convolution refers to x_0. Interpret this relation.

We leave to the reader the task of demonstrating that, if $x_0 = c_0 t$ is reconstructed, one must write

$$\mathcal{E} = \frac{\delta(\Gamma)}{4\pi c_0^3} \quad , \quad \frac{1}{4\pi c_0^2} \iiint_{-\infty}^{\infty} \frac{G(c_0 t - |\mathbf{x} - \mathbf{x}'|, \mathbf{x}')}{|\mathbf{x} - \mathbf{x}'|} d\mathbf{x}'. \tag{85}$$

1.3.5 Riesz Distributions

Let us consider the function

$$\begin{cases} \xi_{R,\alpha} = \dfrac{r^{-1+\alpha}}{2\pi\, 2^{2\alpha}\, \alpha!\,(\alpha-1)!} \,, & x_0 > |x| \,, \\ \xi_{R,\alpha} \equiv 0 \,, & x_0 < |x| \,, \end{cases} \tag{86}$$

where α is a strictly positive real number; then, we can associate with this function the distribution of the same type, defined by

$$\langle \xi_{R,\alpha}, \psi \rangle = \iint d\omega \int_0^1 \frac{(1-r^2)^{-1+\alpha}\, r^2}{2\pi\, 2^{2\alpha}\, \alpha!\,(\alpha-1)!}\, dr \int_0^\infty \phi(\rho, r, \omega)\, \rho^{1+2\alpha}\, d\rho \tag{87}$$

where, in view of the fact that ω is a unit vector, we pose

$$\phi(\rho, r, \omega) = \psi(\rho, \rho r \omega). \tag{88}$$

Let us select, once and for all, the function $\psi(x_0, x)$, which is assumed to be indefinitely differentiable and to be $\equiv 0$ in the neighborhood of infinity (compact base) and let us consider /44

$$T_\alpha(\omega) = \int_0^1 \frac{(1-r^2)^{-1+\alpha}\, r^2}{2\pi\, 2^{2\alpha}\, \alpha!\,(\alpha-1)!}\, dr \int_0^\infty \phi(\rho, r, \omega)\, \rho^{1+2\alpha}\, d\rho \,. \tag{89}$$

We will demonstrate that this can be considered as being an analytic function of the complex variable $\alpha = \alpha' + i\alpha''$ in any complex plane. We leave to the reader the task of demonstrating the following lemma:

Lemma: No matter what the integer N might be, we can write

$$\phi(\rho, r, \omega) = \sum_{n=0}^{N} \rho^n e^{-\rho} \sum_{p=0}^{n} r^p A_{np}(\omega) + \rho^{N+1} \phi_{N+1}(\rho, r, \omega) \tag{90}$$

where $A_{n,p}(\omega)$ is a homogeneous polynomial of the degree p relative to the components ω_1, ω_2, ω_3 of ω, with $\omega_1^2 + \omega_2^2 + \omega_3^2 = 1$ and where ϕ_{N+1} is a function indefinitely differentiable from its arguments, bounded everywhere and tending to zero more rapidly than $e^{-\rho/2}$ as soon as $\rho \to \infty$.

Making use of the formulas

$$\left\{ \begin{aligned} &\int_0^\infty e^{-t} t^{x-1} dt = (x-1)! \\ &\int_0^1 t^{x-1}(1-t)^{y-1} dt = \frac{(x-1)!\,(y-1)!}{(x+y-1)!} \\ &(2x-1)! = 2^{2x} \pi^{-1/2} (x-1)!\,(x-\tfrac{1}{2})! \end{aligned} \right. \tag{91}$$

we obtain

$$\begin{aligned} I_\alpha(\omega) = &\sum_{n=0}^{N} \sum_{p=0}^{n} \frac{(2\alpha+1)(2\alpha+2)\ldots.(2\alpha+n+1)}{(2\alpha+1)(2\alpha+2)\ldots(2\alpha+p+1)} \frac{(p+1)!}{2\pi} A_{n,p}(\omega) \\ &+ \int_0^\infty \frac{\rho^{N+2+2\alpha}}{2\pi\, 2^{2\alpha}\alpha!(\alpha-1)!} d\rho \int_0^1 \phi_{N+1}(\rho, r, \omega)\,(1-r^2)^{-1+\alpha} r^2 dr. \end{aligned} \tag{92}$$

The double sum which occurs in the second term of eq.(92) is obviously an analytical function of α in the entire complex plane. Let us therefore concern ourselves with the double integral; for this, let us consider /45

$$J_{N+1,\alpha}(\rho, \omega) = \frac{1}{(\alpha-1)!} \int_0^1 \phi_{N+1}(\rho, r, \omega)\,(1-r^2)^{-1+\alpha} r^2 dr \tag{93}$$

and demonstrate that this expression is an analytic function of α in the entire complex plane which, in the entire domain bounded by this complex plane, is uniformly augmented by const $xe^{-\rho/2}$. Within the domain of the complex plane in question, we have Re $\alpha > -M$. Since ϕ_{N+1} is indefinitely differentiable, we have

$$\begin{aligned} \phi_{N+1}(\rho, r, \omega) = &\sum_{m=0}^{M} (1-r^2)^m B_{N+1,m}(\rho, \omega) + \\ &+ (1-r^2)^{M+1} \phi_{N+1,M+1}(\rho, r, \omega) \end{aligned} \tag{94}$$

so that B as well as ϕ are augmented by const $xe^{-\rho/2}$. On substituting eq.(94)

in eq.(93), we obtain

$$J_{N+1,\alpha}(\rho,\omega) = \sum_{m=0}^{M} \frac{\alpha(\alpha+1)\cdots\cdots(\alpha+m)}{(\alpha+m+\frac{1}{2})!\,\{(\frac{1}{2})!\}^{-1}} B_{N+1,m}(\rho,\omega) + \frac{1}{(\alpha-1)!}\int_0^1 \phi_{N+1,M+1}(\rho,r,\omega)\,(1-r^2)^{M+\alpha}\, r^2\, dr\,, \tag{95}$$

and it is obvious that the sum is an analytic function of α in the entire complex plane, augmented by const $xe^{-\rho/2}$ in Re $\alpha > -M$ and that the same property is present for the product formed by the integral and $1/(\alpha - 1)!$, from which it follows that $J_{N+1,\alpha}(\rho, \omega)$ exhibits many of the indicated properties. Finally, since the repetitive integral which occurs in the second term of eq.(92) is equal to $\int_0^\infty \frac{\rho^{N+2+2\alpha} J_{N+1,\alpha}(\rho, \omega)}{2n2^{2\alpha}\alpha!}\, d\rho$, this represents an analytic function of α /46 for Re $\alpha > -1 - N/2$. However, N is arbitrary so that $I_\alpha(\omega)$ is an analytic function of α in the entire complex plane. For each α, this is an indefinitely differentiable function of ω on a sphere of radius 1. As a consequence,

$$\langle \xi_{R,\alpha}, \psi \rangle = \iint I_\alpha(\omega)\, d\omega\,, \tag{96}$$

is an analytic function of α in the entire complex plane.

Theorem 13: There exists a distribution $\mathfrak{S}_{R,\alpha}$ known as the Riesz distribution, defined in the following manner: For Re $\alpha > 0$, this distribution is equal to the function

$$\xi_{R,\alpha} = \begin{cases} \dfrac{\rho^{-1+\alpha}}{2n2^{2\alpha}\alpha!(\alpha-1)!}\,, & x_0 \geq * \\ 0\,, & x_0 < * \end{cases} \tag{97}$$

Its definition is extended to the overall complex plane α by the statement that the analytic function of α, defined in Re $\alpha > 0$ from eq.(97) by $I(\alpha) = \langle \mathfrak{S}_{R,\alpha}\rho\psi \rangle$ can be extended, for each indefinitely differentiable function ψ on a compact base, to the entire complex plane.

Let us assume Re $\alpha > 2$ and that $\mathfrak{S}_{R,\alpha}$ is twice continuously differentiable, so that

$$\left(\frac{\partial^2}{\partial x_0^2} - \Delta\right) \xi_{R,\alpha} = \xi_{R,\alpha-1} \tag{98}$$

as also results from a direct calculation. For Re $\alpha > 2$, eq.(98) is an equality between functions; however, according to the above, this will be an equality /47 between distributions, for any α in the complex plane.

We leave to the reader the task of proving [by using eq.(87)] the following theorem.

Theorem 14: Given are the following identities:

$$\xi_{R,0} = \frac{\delta(\Gamma)\ 1(x_0)}{2\pi}, \tag{99}$$

$$\xi_{R,-1} = \delta(x_0)\,\delta(x_1)\,\delta(x_2)\,\delta(x_3). \tag{100}$$

Problem 10: Calculate the product of convolution (on x) $\mathfrak{S}_{R,\alpha} * F(x)$ at Re $\alpha > 0$, and demonstrate that its limit, at $\alpha \to 0$, exists and is equal to $x_0\mathfrak{M}^x_{x_0}(F)$. Interpret the result.

Note: M.Riesz developed his theory before the theory of distributions had been established, for which he defined $\mathfrak{S}_{R,\alpha} * F$ at Re $\alpha > 0$ by integration in the ordinary sense; he extended the definition by analytic extrapolation, using the hypotheses of differentiability on F. In other words, he gave a sense to $\mathfrak{S}_{R,\alpha}$, for any complex α, by analytic prolongation. The mathematician L.Schwartz gave $\mathfrak{S}_{R,\alpha}$ a sense, for any α, by means of the theory of distributions. The connection between the two viewpoints is expressed by the following formula:

$$\langle [\xi_{R,\alpha} * F](x_0, x), \varphi(x_0, x)\rangle = \langle F(x_0, x), \langle \xi_{R,\alpha}(y_0, y), \varphi(x_0 + y_0, x + y)\rangle\rangle. \tag{101}$$

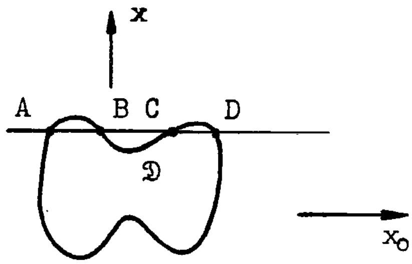
It can be stated that M.Riesz analytically extended the left-hand term whereas L.Schwartz analytically extended the right-hand term.

1.3.6 Basic Formula /48

The number of dimensions of space is indifferently 1, 2, or 3. Thus, $\Delta = \Delta \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$ and n = 1, 2, or 3. We pose $\mathfrak{L} = \frac{\partial^2}{\partial x_0^2} - \Delta$. Let U and V be twice continuously differentiable functions, yielding

$$U\mathfrak{L}V - V\mathfrak{L}U = \frac{\partial}{\partial x_0}\left(U\frac{\partial V}{\partial x_0} - V\frac{\partial U}{\partial x_0}\right) - \nabla\cdot\left(U\nabla V - V\nabla U\right), \tag{102}$$

with the notation $\nabla \cdot (U \nabla V) = \sum_{i=1}^{n} \frac{\partial}{\partial x_i}\left(U \frac{\partial V}{\partial x_i}\right)$. Let us integrate the identity (102) in a domain $\mathfrak{D}$ of n + 1 dimensions, limited by a boundary Σ to



n dimensions. Let us denote $S(x_0)$, the cross-sectional surface of Σ, by $x_0 =$ = const and denote $\mathfrak{D}(x_0)$, the cross section of $\mathfrak{D}$, by x_0 = const. We then have

$$\int_{\mathfrak{D}} (U \mathcal{L} V - V \mathcal{L} U)\, dx_0\, d\mathbf{x} = \int_{\Sigma} \left(U \frac{\partial V}{\partial x_0} - V \frac{\partial U}{\partial x_0}\right) \varepsilon\, d\mathbf{x} - \int\int_{S(x_0)} \left(U \frac{dV}{dn} - V \frac{dU}{dn}\right) dx_0\, dS \,, \tag{103}$$

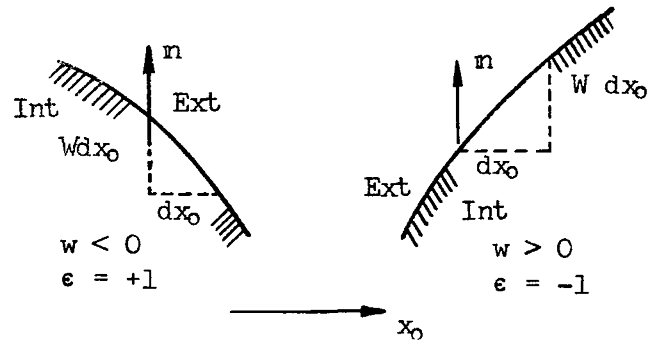
where the notations are as follows: $\frac{d}{dn}$ is the derivative normal to $S(x_0)$ pointing toward the exterior of $\mathfrak{D}(x_0)$; ds is the area element (without sign) of $S(x_0)$; $d\,\mathbf{x}$ is the volume element of Σ in orthogonal mapping onto x_0; ϵ has a value of +1 or −1 and, for example in the case of the accompanying diagram, $\epsilon_A = -1$, $\epsilon_B = +1$, $\epsilon_C = -1$, $\epsilon_D = +1$. We used the terminology corresponding to n = 3 and leave to the reader the task of transcribing this for the case of n = = 2 and n = 1.

Let $\mathbf{n}$ be the unit vector normal to $S(x_0)$ in x_0 = const, pointing toward the exterior of $\mathfrak{D}(x_0)$. Let us define the (algebraic) rate of normal displacement of $S(x_0)$ by the condition that, if $\mathbf{x}$ is located on $S(x_0)$, then $\mathbf{x} + w\,\mathbf{n}\,dx_0$ is /49
located on $S(x_0 + dx_0)$ as indicated in the accompanying diagram; we will then have

$$\varepsilon\, d\mathbf{x} = -\frac{w\, d\Sigma}{\sqrt{1+w^2}}, \quad dx_0\, dS = \frac{d\Sigma}{\sqrt{1+w^2}}, \tag{104}$$

if $d\Sigma$ denotes the area element (without sign) of Σ. Let us next consider the vector

$$\mathfrak{D} = \left(\frac{w}{\sqrt{1+w^2}}, \frac{n}{\sqrt{1+w^2}}\right) = (\nu_0, \nu_i) \qquad i = 1, 2, n \tag{105}$$

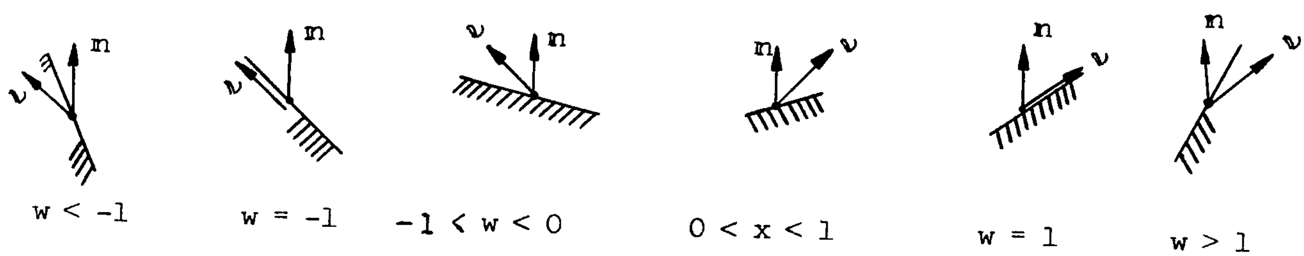


and define the conormal derivative, exterior to Σ, by

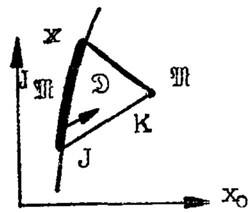
$$\frac{d}{d\nu} = \nu_0 \frac{\partial}{\partial x_0} + \sum_{i=1}^{n} \nu_i \frac{\partial}{\partial x_i} = \frac{w}{\sqrt{1+w^2}} \frac{\partial}{\partial x_0} + \frac{1}{\sqrt{1+w^2}} \frac{d}{dn} . \tag{106}$$

Equation (103) is then brought to the following form:

$$\int_{\mathfrak{D}} (U \mathcal{L} V - V \mathcal{L} U)\, dx_0\, dx + \int_{\Sigma} \left(U \frac{dV}{d\nu} - V \frac{dU}{d\nu}\right) d\Sigma = 0 \tag{107}$$



The accompanying diagram indicates the various positions of the exterior conormal, depending on the value of w. It is obvious that, for $|w| = 1$, the conormal is tangent to Σ. The hatching indicates the side on which the interior of $\mathfrak{D}$ is located. A particular case of interest for the application of eq.(107) is that of the next diagram which, if U or V vanish on K, is singularly charac-



terized by w = -1. Equation (107) will then assume the form

$$\int_{\mathfrak{D}} (U\mathcal{L}V - V\mathcal{L}U)\, dx_0\, dx + \int_{J_{\mathfrak{M}}} \left(U\frac{dV}{d\nu} - V\frac{dU}{d\nu}\right) d\Sigma = 0 \,. \tag{108}$$

Let us now make an application by assuming that /50

$$V = \frac{\Gamma^{\frac{1-n}{2}+\alpha}}{H_n(\alpha)} = \mathcal{E}_{AR,\alpha}\,, \qquad \Gamma = (x_0-y_0)^2 - \sum_{i=1}^{n}(x_i-y_i)^2\,, \tag{109}$$
$$H_n(\alpha) = \pi^{\frac{n-1}{2}}\, 2^{2\alpha+1}\left(\alpha+\frac{1-n}{2}\right)!\;\alpha!\,,$$

where (y_0, $\mathbf{y}$) define the moving point in $\mathfrak{D}$, and where (x_0, $\mathbf{x}$) define the point $\mathfrak{M}$. Let, on the other hand, U be an arbitrary twice continuously differentiable function; if Re α $\frac{n+1}{2}$, then V is twice continuously differentiable and vanishes on K, so that eq.(108) can be applied. Taking into consideration that, with the new definition of $\mathfrak{E}_{AR,\alpha}$, eq.(98) is valid no matter what the number of dimensions n of space (ordinary) might be, we will obtain

$$\int_{\mathfrak{D}} \left\{U\,\mathcal{E}_{AR,\alpha-1} - \mathcal{L}U\cdot\mathcal{E}_{AR,\alpha}\right\} dy_0\, dy + \int_{J_{\mathfrak{M}}} \left(U\frac{d}{d\nu}\mathcal{E}_{AR,\alpha} - \frac{dU}{d\nu}\mathcal{E}_{AR,\alpha}\right) d\Sigma = 0 \tag{110}$$

For defining $\mathfrak{J}_{\mathfrak{M}}$, let us refer to the diagram on the preceding page. If $\mathfrak{M}$ as well as $\mathfrak{J}$ and U are fixed, the first term will depend only on α, being a sum of terms each of which is analytical for Re $\alpha > \frac{n+1}{2}$ and whose sum is ≡ 0 under the same conditions. This sum remains ≡ 0 in the entire domain of the complex

plane α, where each term can be analytically extended. We leave to the reader the task to demonstrate, by making use of the technique described in Sect.1.3.5, that

$$\mathop{\mathrm{Prol}}_{\alpha=-1} \int_{\mathcal{D}} U\, \mathcal{E}_{AR,\alpha}\, dy_0\, dy = U(\mathcal{M}) - U(x_0, x). \tag{111}$$

This will yield the following formula:

$$U(\mathcal{M}) = \mathop{\mathrm{Prol}}_{\alpha=0} \left\{ \int_{\mathcal{D}} \frac{\Gamma^{\frac{1-n}{2}+\alpha}}{H_n(\alpha)} \mathcal{L}\, U(y_0,y)\, dy_0\, dy \; + \right.$$
$$\left. + \int_{\mathfrak{S}_{\mathcal{M}}} \left(\frac{dU}{d\nu}(y_0,y)\, \frac{\Gamma^{\frac{1-n}{2}+\alpha}}{H_n(\alpha)} - U(y_0,y)\, \frac{d}{d\nu}\, \frac{\Gamma^{\frac{1-n}{2}+\alpha}}{H_n(\alpha)} \right) d\Sigma \right\}. \tag{112}$$

When $\mathfrak{S}$ is $x_0 = 0$ and $n = 3$, it is relatively easy to write the second term of eq.(112) in explicit form; the reader is advised to do this and thus to return to eq.(55). To avoid unnecessary research, we mention here the artifice /51
which consists in writing $\int_{\mathfrak{S}_{\mathcal{M}}} U \frac{d}{d\nu} \mathfrak{E}_{AR,\alpha} d\Sigma = \frac{d}{d\nu} \int U \mathfrak{E}_{AR,\alpha} d\Sigma$ (by means of a suitable convention!). We will be satisfied here with considering the case $n = 2$ when $\mathfrak{S}$ is $x_0 = 0$. The analytic prolongation is immediate because of the above-mentioned artifice (note that $\frac{d}{d\nu}$ refers to y_a!).

Theorem 15: The function

$$\phi(x_0,x,y) = \frac{1}{2\pi} \int_0^{x_0} dx'_0 \iint_{(x-x')^2+(y-y')^2<(x_0-x'_0)^2} \frac{G(x'_0,x',y')}{\sqrt{(x_0-x'_0)^2-(x-x')^2-(y-y')^2}}\, dx'\, dy'$$
$$+ \frac{1}{2\pi} \iint_{(x-x')^2+(y-y')^2<x_0^2} \frac{F_1(x',y')}{\sqrt{x_0^2-(x-x')^2-(y-y')^2}}\, dx'\, dy' \tag{113}$$

$$+\frac{1}{2\pi}\frac{\partial}{\partial x_0}\iint\limits_{(x-x')^2+(y-y')^2<x_0^2}\frac{F_0(x',y')}{\sqrt{x_0^2-(x-x')^2-(y-y')^2}}\,dx'\,dy' \quad ,$$

proves

$$\left\{\begin{aligned} \frac{\partial^2\phi}{\partial x_0^2}-\frac{\partial^2\phi}{\partial x^2}-\frac{\partial^2\phi}{\partial y^2} &= G(x_0,x,y)\,, \qquad x_0>0 \\ \phi(0,x,y) &= F_0(x,y)\,, \\ \frac{\partial\phi(0,x,y)}{\partial x_0} &= F_1(x,y)\,, \end{aligned}\right. \tag{114}$$

if, for example G and F_1 are twice continuously differentiable and if F_0 is three times continuously differentiable.

Equation (112) guarantees that, if there exists a twice continuously differentiable function ϕ in $x_0 > 0$ and continuously differentiable in $x_0 \geq 0$, then it will be given by eq.(113). In other words, we have defined the form of the solution and it is only necessary to obtain the proof. For this, we return /52 to eq.(112) and note ϕ_α, which is the function obtained by posing $\Omega U = G$, $\frac{dU}{d\nu} = F_1$, $U = F_0$ in the second term, before having performed the analytic prolongation. For a sufficiently large Re α, it is not at all difficult to make a repeated differentiation under the integral sign. It is then sufficient to form the first terms of eq.(114), relative to ϕ_α, and to perform the analytic prolongation up to $\alpha = 0$ so as to obtain the desired proof, provided that such prolongation is at all possible. The considerations in Section 1.3.5 demonstrate that this necessitates hypotheses of differentiability for F_0, F_1, G; those that we indicated in the formulation of the theorem are not the weakest possible, but we cannot further go into this relatively delicate question.

Let us now attempt to write the last term of eq.(113) in a more explicit form; we thus will write

$$F_0(x+r\cos\theta,\ y+r\sin\theta) = \bar{F}_0(r,\theta;x,y)\,, \tag{115}$$

in such a manner that we have to calculate

$$\frac{1}{2\pi}\frac{\partial}{\partial x_0}\int_0^{2\pi}d\theta\int_0^{x_0}\frac{\bar{F}_0(r,\theta;x,y)\,r\,dr}{\sqrt{x_0^2-r^2}} =$$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\theta \frac{\partial}{\partial x_0}\int_0^{x_0} \frac{\bar{F}_0(x_0,\theta;x,y)}{\sqrt{x_0^2-r^2}}\, r\,dr - \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^{x_0} \frac{\frac{\partial \bar{F}_0(x_0,\theta;x,y)}{\partial x_0}}{\sqrt{x_0^2-r^2}}\, r\,dr \tag{116}$$

$$- \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^{x_0} \frac{x_0}{(x_0^2-r^2)^{3/2}} \left\{ \bar{F}_0(r,\theta;x,y) - \bar{F}_0(x_0,\theta;x,y) \right\} r\,dr$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \bar{F}_0(x_0,\theta;x,y)\, d\theta - \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^{x_0} \frac{x_0}{(x_0^2-r^2)^{3/2}} \left\{ \bar{F}_0(r,\theta;x,y) - \bar{F}_0(x_0,\theta;x,y) \right\} r\,dr .$$

Let us pick up this result in a different manner: Let us assume that, returning to eq.(112), we did not permute $\frac{d}{d\nu}$ and $\int$, which means that we must calculate $\underset{\alpha=0}{\text{Prol}}\, I_\alpha$, with

$$I_\alpha = \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^{x_0} r\,dr\ \bar{F}_0(r,\theta;x,y) \frac{\partial}{\partial x_0} \frac{(x_0^2-r^2)^{\alpha-1/2}}{H_2(\alpha)} . \tag{117}$$

For this purpose, let us denote by $\int^r \frac{\partial}{\partial x_0} \left(\frac{(x_0^2 - r^2)^{\alpha-\frac{1}{2}}}{H_2(\alpha)} \right) r\,dr$, the primitive in r of the integrand, so that we obtain /53

$$I_\alpha = \frac{1}{2\pi}\int_0^{2\pi} \bar{F}_0(x_0,\theta;x,y) \left\{ \int^r \frac{\partial}{\partial x_0}\left(\frac{(x_0^2-r^2)^{\alpha-\frac{1}{2}}}{H_2(\alpha)} \right) r\,dr \right\}_{r=0}^{r=x_0} d\theta$$

$$+ \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^{x_0} r\,dr \left\{ \bar{F}_0(r,\theta;x,y) - \bar{F}_0(x_0,\theta;x,y) \right\} \frac{\partial}{\partial x_0} \frac{(x_0^2-r^2)^{\alpha-1/2}}{H_2(\alpha)} , \tag{118}$$

and, by making $\alpha \to 0$, we return to eq.(116). Finally, let us apply Hadamard's finite partial integration technique. We then write

$$\left\{ \quad I(x_0,\varepsilon) = \frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^{x_0-\varepsilon} \frac{\bar{F}_0(r,\theta;x,y)}{\sqrt{x_0^2-r^2}}\, r\,dr , \right. \tag{119}$$

$$J(x_0,\varepsilon) = -\frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^{x_0-\varepsilon} \frac{x_0 \; \bar{F}_0(r,\theta;x,y)}{(x_0^2-r^2)^{3/2}}\, r\, dr ,$$

and note that

$$\frac{\partial I(x_0,\varepsilon)}{\partial x_0} = J(x_0,\varepsilon) + \frac{x_0-\varepsilon}{\sqrt{\varepsilon(2x_0-\varepsilon)}} \quad \frac{1}{2\pi}\int_0^{2\pi} d\theta \;\; \bar{F}_0(x_0-\varepsilon,\theta;x,y), \tag{120}$$

so that we obtain

$$\begin{aligned} J_0(x_0,\varepsilon) = &-\frac{1}{2\pi}\int_0^{2\pi}\left\{\frac{x_0}{\sqrt{\varepsilon(2x_0-\varepsilon)}} - 1\right\} \bar{F}(x_0,\theta;x,y)\, d\theta \\ &-\frac{1}{2\pi}\int_0^{2\pi} d\theta \int_0^{x_0-\varepsilon} \frac{x_0[\bar{F}_0(r,\theta;x,y)-\bar{F}_0(x_0,\theta;x,y)]}{(x_0^2-r^2)^{3/2}}\, r\, dr \end{aligned} \tag{121}$$

from which eq.(116) results. Hadamard wrote

$$\begin{aligned} \frac{\partial}{\partial x_0}\int_0^{x_0} \frac{F(r)}{\sqrt{x_0^2-r^2}}\, r\, dr &= \overline{\int_0^{x_0}} r\, F(r)\, \frac{\partial}{\partial x_0}\left(\frac{1}{\sqrt{x_0^2-r^2}}\right) dr \\ &= \lim_{\varepsilon\to 0}\left\{\int_0^{x_0-\varepsilon} r\, F(r)\, \frac{\partial}{\partial x_0}\left(\frac{1}{\sqrt{x_0^2-r^2}}\right) dr - \varepsilon^{-1/2} A\right\} \end{aligned} \tag{122}$$

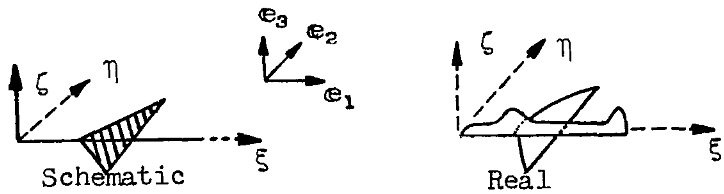
where A is precisely such that the limit exists. In this manner, Hadamard gives a sense to the divergent integral which is contained in the second term, by subtraction from fractional infinity.

1.4 Application to the Calculus of the Acoustic Field Produced by the Flight of an Aircraft /54

1.4.1 Schematization of the Aircraft

From the geometric viewpoint, the aircraft is reduced to its skeleton: The fuselage is reduced to its axis and the wing to its planform, as indicated in the accompanying diagram. In addition, for the fuselage we also give the law

of areas of the cross sections, i.e., $\delta(\xi)$ and the law of stresses $\rho_0\bar{\omega}_F(\xi, t)$, which means that, although the fuselage is reduced to its axis from the geometric viewpoint, its acoustic effect on the ambient air depends on $S(\xi)$ and $\bar{\omega}_F(\xi, t)$



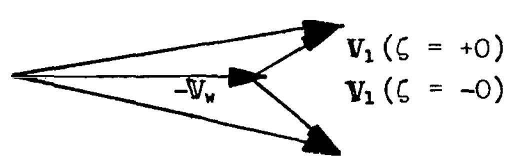
in a manner which is precisely the item to be determined. The meaning of $S(\xi)$ is obvious. For $\bar{\omega}_F(\xi, t)$ we assume that the force exerted by the air on the fuselage is a force concentrated along the axis with a density $\rho_0\omega_F$, which means that the force exerted on the section of the fuselage, comprised between ξ_1 and ξ_2, is equal to $\int_{\xi_1}^{\xi_2} \rho_0\bar{\omega}_F(\xi, t)d\xi$, at the instant t.

With respect to the wing, we characterize it by the law of thicknesses $h(\xi, \eta)$ and by the law of stresses $\rho_0\bar{\omega}_W(\xi, \eta, t)\,\mathbf{e}_3(t)$ where $\mathbf{e}_3(t)$ denotes the unit vector normal to the plane of the skeleton. This is to express that, if the wing from the geometric viewpoint is reduced to its planform, then the acoustic field created by the wing obeys, in a still to be determined manner, the law of thicknesses and the law of stresses. The significance is as follows for ω_W: The stress exerted by the air on the segment A of the wing surface /55
is $\iint \rho_0\bar{\omega}_W(\xi, \eta, t)\,\mathbf{e}_3\,ds$, at the instant t.

Except with respect to the problems of aeroelasticity which we disregard here, the motion of the aircraft is that of a solid body and is defined by the velocity vector of the center of gravity $-V_W(t)\,\mathbf{e}_1(t)$ and by the angular velocity vector $\boldsymbol{\Omega}(t)$.

1.4.2 Fundamental Equation

On the wing, the velocity vector of the fluid is tangent to the surface of the latter, on the pressure side (exposed to the wind) as well as on the suction



side. In aircraft-fixed axes (axes ξ, η, ζ), the modulus of the fluid velocity vector is very close to V_W while its direction is subject to a discontinuity

having a slope equal to $\frac{\partial h}{\partial \xi}$. Thus taking into consideration $[f] = f_{\zeta=+0} - f_{\zeta=-0}$, which is the discontinuity of f on traversing the plane $\zeta = 0$, we will finally obtain

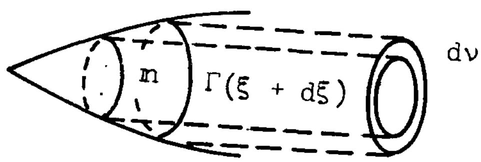
$$[\mathbb{V}_1 \cdot \mathbb{e}_3] = V_w \frac{\partial h}{\partial \xi}, \tag{1}$$

using the notations $\mathbb{V}_1$, p_1, ρ_1 of the beginning of this Chapter.

With respect to the fuselage, assuming that $\mathbb{n}$ denotes the normal unit vector directed toward the exterior, we will have – on the surface – /56

$$\mathbb{V}_1 \cdot \mathbb{n} = V_w \frac{d\nu}{d\xi}, \tag{2}$$

denoting by $d\nu$ the distance between the projections, onto a plane normal to the axis of the fuselage section contours $\Gamma(\xi)$ and $\Gamma(\xi + d\xi)$, by the planes ξ and



$\xi + d\xi$. If $\phi(\xi, \eta, \zeta)$ is a base function and if the wing is disregarded, we will thus have

$$-\iiint_{\mathcal{D}_e} \mathbb{V}_1 \cdot \nabla \phi \, d\xi \, d\eta \, d\zeta = \iiint_{\mathcal{D}_e} \phi \, \nabla \cdot \mathbb{V}_1 \, d\xi \, d\eta \, d\zeta + \int d\xi \int_{\Gamma(\xi)} V_w \frac{d\nu}{d\xi} \phi_F \, ds, \tag{3}$$

denoting by ϕ_F the value of ϕ on the fuselage and by $\mathcal{D}_e$ the space exterior to the fuselage. If the fuselage is schematized by an axis on which a law of areas is superimposed, eq.(3) must be written in the form of

$$\langle \nabla \cdot \mathbb{V}_1, \phi \rangle = \iiint \phi \, \nabla \cdot \mathbb{V}_1 \, d\xi \, d\eta \, d\zeta + \int \phi(\xi, 0, 0) \, d\xi \int_{\Gamma(\xi)} V_w \frac{d\nu}{d\xi} \, ds = \tag{4}$$

$$= \iiint \phi \, \bar{\nabla}\cdot \mathbb{V}_1 \, d\xi \, d\eta \, d\zeta + \int V_w \frac{ds(\xi)}{d\xi} \phi(\xi, 0, 0) \, d\xi .$$

On combining eqs.(1) and (4), it is obvious that we can write

$$\bar{\nabla}\cdot \mathbb{V}_1 = (\nabla \cdot \mathbb{V}_1) + V_A \frac{\partial h}{\partial \xi} \delta(\zeta) + V_w \frac{ds}{d\xi} \delta(\eta) \delta(\zeta), \tag{5}$$

where the first term designates the distribution divergence while $(\nabla \cdot \mathbb{V}_1)$ denotes the function divergence.

Let us now attempt to interpret $\bar{\omega}_F$ and $\bar{\omega}_w$. With respect to $\bar{\omega}_w$, it is immediately obvious that /57

$$[p_1] = -\rho_0 \varpi_w . \tag{6}$$

With respect to $\bar{\omega}_F$, neglecting the wing, we obtain

$$-\iiint p_1 \nabla \phi \, d\xi \, d\eta \, d\zeta = \iiint_{\mathcal{D}_e} \phi \, \nabla p_1 \, d\xi \, d\eta \, d\zeta + \int d\xi \int_{\Gamma(\xi)} \phi_F \, p_{1_F} \, n \, ds, \tag{7}$$

and, if the fuselage is schematized by an axis and a law of stress, we must write

$$\langle \phi, \bar{\nabla} p_1 \rangle = \iiint_{\mathcal{D}} \phi \, \nabla p_1 \, d\xi \, d\eta \, d\zeta - \int \rho_0 \varpi_F(\xi, t) \, \phi(\xi, 0, 0) \, d\xi, \tag{8}$$

which will yield

$$\bar{\nabla} p_1 = (\nabla p_1) - \rho_0 \varpi_w \mathbf{e}_3 \delta(\zeta) - \rho_0 \varpi_F \delta(\eta) \delta(\zeta), \tag{9}$$

where, again, the first term denotes the distribution gradient while (∇p_1) denotes the function gradient.

In concluding our study, we can - in agreement with the schematization defined above - disregard the presence of the aircraft if we treat the acoustic field as a distribution and if we replace the acoustic equations, in a homogeneous medium, by

$$\begin{cases} \frac{\partial \rho_1}{\partial t} + \rho_0 \nabla \cdot \mathbb{V}_1 = \rho_0 V_w(t) \frac{\partial h(\xi,\eta)}{\partial \xi} \delta(\zeta) + \rho_0 V_w(t) \frac{d S(\xi)}{d\xi} \delta(\eta)\, \delta(\zeta), & a) \\ \rho_0 \frac{\partial \mathbb{V}_1}{\partial t} + \nabla p_1 = -\rho_0\, \varpi_w(\xi,\eta,t)\, \mathbb{e}_3(t)\, \delta(\zeta) - \rho_0\, \varpi_F(\xi,\eta,t)\, \delta(\eta)\, \delta(\zeta), & b) \\ S_1 = 0\,. & c) \end{cases} \tag{10}$$

It will be noted that the second term is expressed explicitly in axes fixed with respect to the aircraft while the first term is written in generally fixed axes. /58

Let us now, by means of eqs.(10), resume the analysis which had previously yielded the wave equation. For this purpose, we note that we still have

$$\nabla p_1 = c_0^2\, \nabla \rho_1\,, \tag{11}$$

and that, consequently, the following equation is obtained for p_1:

$$\frac{1}{c_0^2}\frac{\partial^2 p_1}{\partial t^2} - \Delta p_1 = \frac{\partial A}{\partial t} + \nabla \cdot \mathbb{B}\,, \tag{12}$$

if we denote by A and $-\mathbb{B}$ the second terms of eqs.(10a) and (10b). It should be noted that $\frac{\partial A}{\partial t}$ represents the temporal derivative, evaluated in fixed axes, whereas A, naturally, is expressed in moving axes; if $\frac{\partial A}{\partial \tau}$ is the temporal derivative in moving axes, we have

$$\frac{\partial A}{\partial t} = \frac{\partial A}{\partial \tau} + V_w \frac{\partial A}{\partial \xi}\,. \tag{12}$$

Finally, this will lead to

$$\frac{1}{c_0^2}\frac{\partial^2 p_1}{\partial t^2} - \Delta p_1 = \rho_0 c_0^2\, \mathcal{T}\,, \tag{13}$$

where $\mathcal{T}$ is a distribution made over the framework of the aircraft, i.e.,

$$\rho_0 c_0^2\, \mathcal{T} = \rho_0 \left(V_w^2(t) \frac{\partial^2 h(\xi,\eta)}{\partial \xi^2} + \frac{dV_w(t)}{dt}\frac{\partial h(\xi,\eta)}{\partial \xi} \right) \delta(\zeta) \tag{14}$$

$$+\rho_0\,\overline{\omega}_w(\xi,\eta,t)\,\frac{\partial\,\delta(\zeta)}{\partial\zeta}+$$

$$+\rho_0\left(V_w^2(t)\,\frac{d^2s(\xi)}{d\xi^2}+\frac{dV(t)}{dt}\,\frac{ds(\xi)}{d\xi}\right)\delta(\eta)\,\delta(\zeta)\;+$$

$$+\rho_0\,\overline{\omega}_F(\xi,t)\cdot\nabla\left(\delta(\eta)\,\delta(\zeta)\right).$$

To avoid ambiguity, let us define that, if ϕ $(\xi,\ \eta,\ \zeta,\ \tau)$ denotes a base function, we will have /59

$$\langle\rho_0\,c_0^2\;\tau,\phi\rangle=\iiint d\tau\,d\xi\,d\eta\left\{\left(\rho_0V_w^2(\tau)\,\frac{\partial^2h(\xi,\eta)}{\partial\xi^2}+\rho_0\,\frac{dV_w(\tau)}{d\tau}\,\frac{\partial h(\xi,\eta)}{\partial\xi}\right)\phi_{\zeta=0}\right.$$

$$\left.-\rho_0\,\overline{\omega}_w(\xi,\eta,\tau)\,\frac{\partial\phi(\xi,\eta,0,\tau)}{\partial\zeta}\right\}+$$

$$+\iint d\tau\,d\xi\left\{\left(\rho_0V_w^2(\tau)\,\frac{d^2s(\xi)}{d\xi^2}+\rho_0\,\frac{dV_w(\tau)}{d\tau}\,\frac{ds(\xi)}{d\xi}\right)\phi(\xi,0,0,\tau)\right.\qquad(15)$$

$$\left.-\rho_0\,\overline{\omega}_F(\xi,\tau)\cdot\nabla\phi(\xi,0,0,\tau)\right\}.$$

We have explicitly assumed that h and s are time-invariant, since this is sufficient to cover all practical cases; however, in principle, there is no objection to re-establishing a time dependence. By association of ideas, if L denotes the length of the aircraft and ng its acceleration (g being the acceleration of gravity), we will have

$$\frac{\left|\dfrac{dV_w}{dt}\,\dfrac{ds}{d\xi}\right|}{\left|V_w^2\,\dfrac{d^2s}{d\xi^2}\right|}=O\left(\frac{n}{10}\,\frac{L}{10}\,\frac{1}{Ma^2}\,10^{-2}\right)\qquad(16)$$

where L is expressed in meters and Ma is the Mach number of flight. Except possibly in the takeoff phase or in the case of a very low-speed aircraft, the terms in $\frac{dV_w}{dt}$ in the expression of Σ can be neglected.

In the present formulation, it is obvious that no limit condition need be written on the aircraft (such conditions would have to be written for sea level while for a flight - for example - of the skimming or hedgehopping type, we will neglect them). It would be useful to consider initial conditions, but they can also be disregarded if the motion is investigated from its very beginning. /60

Theorem 16 (theorem of applied mathematics!): In a homogeneous atmosphere, without wind, the acoustic field produced by the flight of an aircraft is defined by a perturbation pressure field which is the distribution solution of the equation

$$\frac{1}{c_0^2}\frac{\partial^2 p_1}{\partial t^2} - \Delta p_1 = \rho_0 c_0^2 \tau , \tag{17}$$

stipulating that the radiant solution is involved here. The distribution τ is carried by the framework of the aircraft and reads as follows:

$$\begin{aligned}\rho_0 c_0^2 \tau = {} & \rho_0 V_w^2(t) \frac{\partial^2 h(\xi,\eta)}{\partial \xi^2}\delta(\zeta) + \rho_0 \varpi_w(\xi,\eta,t)\frac{\partial \delta(\zeta)}{\partial \zeta} + \\ & + \rho_0 V_w^2(t)\frac{d^2 S(\xi)}{d\xi^2}\delta(\eta)\delta(\zeta) + \rho_0 \varpi_F(\xi,t)\cdot\nabla\delta(\zeta)\delta(\eta) ,\end{aligned} \tag{18}$$

in axes moving with the aircraft skeleton, consisting of a fuselage reduced to the axis $\eta = \zeta = 0$ with the law of areas of the cross sections $S(\xi)$ and the law of lift $-\rho_0\varpi_F(\xi, \eta, t)$ and of a wing flattened in the plane $\zeta = 0$ with the law of thicknesses $h(\xi, \eta)$ and the law of lift $-\rho_0\varpi_w(\xi, \eta, t)\, e_3(t)$ where $e_3(t)$ is the unit vector of the axis ζ. The coordinates ξ, η, ζ are rectangular and fixed with respect to the aircraft. Since the pressure field is known, the field of volume mass and the velocity field are obtained in virtue of

$$\begin{cases} \rho_1 = c_0^{-2} p_1 , \\ \rho_0 \dfrac{\partial V_1}{\partial t} + \nabla p_1 = -\rho_0 \varpi_w e_3 \delta(\zeta) - \rho_0 \varpi_F \delta(\eta)\delta(\zeta) . \end{cases} \tag{19}$$

1.4.3 Determination of the Acoustic Field /61

The solution is furnished by theorems 12 and 16, in the form of

$$p_1 = \rho_0 c_0^2 \tau * \frac{1(t)\,\delta(\Gamma)}{2\pi c_0} , \tag{20}$$

since

$$\frac{1}{c_0^2}\left(\frac{\partial^2}{\partial t^2}-\Delta\right)\left\{\rho_0 c_0^2 \tau * \frac{1(t)\,\delta(\Gamma)}{2\pi c_0}\right\} = \rho_0 c_0^2 \tau * \left(\frac{1}{c_0^2}\frac{\partial^2}{\partial t^2}-\Delta\right)\frac{1(t)\,\delta(\Gamma)}{2\pi c_0} \tag{21}$$

$$= \rho_0 c_0^2 \tau * \delta(t)\,\delta(x)\,\delta(y)\,\delta(z) = \rho_0 c_0^2 \tau,$$

and since $\frac{1(t)\delta(\Gamma)}{2\pi c_0}$ is radiant. It then remains to write in explicit form the convolution entering eq.(20). Let us use the notation

$$\mathfrak{M} = (t, \mathbb{M}), \tag{22}$$

for denoting an (instant - point) event by defining $t_{\mathfrak{M}}$, if such is necessary. Let $\phi(\mathfrak{M})$ be a base function and let $\underset{\sim}{P}$ be a current event, so that eq.(20) will be written in the form

$$<p_1(\mathfrak{M}), \phi(\mathfrak{M})> = \ll \rho_0 c_0^2 \tau(\mathfrak{P}) \frac{1(t_{\mathfrak{M}}-t_{\mathfrak{P}})\,\delta[\Gamma(\mathfrak{M}-\mathfrak{P})]}{2\pi c_0}, \phi(\mathfrak{M})\gg, \tag{23}$$

where $\ll \; \gg$ denotes the operation of writing in explicit form a distribution in space of the eight variables $\underset{\sim}{P}$, $\mathfrak{M}$. The best way for performing the calculation is to use fixed axes $(t, x, y, z) = (t, \boldsymbol{x})$ for $\mathfrak{M}$ and moving axes for $\underset{\sim}{P} = (\tau, \xi, \eta, \zeta) = (\tau, \boldsymbol{\xi})$. This will yield

$$\Gamma(\mathfrak{M}-\mathfrak{P}) = (t-\tau)^2 - |\boldsymbol{x} - \boldsymbol{x}_0(\tau) - \xi \boldsymbol{e}_1(\tau) - \eta \boldsymbol{e}_2(\tau) - \zeta \boldsymbol{e}_3(\tau)|^2 \tag{24}$$

denoting by $\boldsymbol{x}_0(\tau)$ the position, in fixed axes, of the origin $\xi = \eta = \delta = 0$ of the entrained axes. Note that we have

$$\mathbb{V}_A(\tau) = \frac{d\,\boldsymbol{x}_0(\tau)}{d\tau}. \tag{25}$$

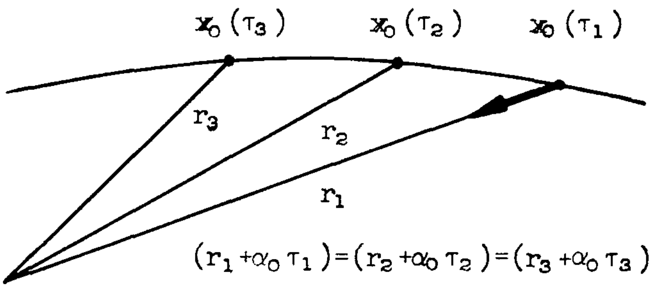
Let us denote by τ_i the roots of the equation in τ, $\Gamma = 0$, for fixed t, $\boldsymbol{x}$, $\boldsymbol{\xi}$. /62
The accompanying diagram shows a case in which three roots are present. Posing $r(\tau) = \boldsymbol{x} - \boldsymbol{x}_0(\tau)$, the function $c_0\tau + r = s(\tau)$ assumes the same values for the three roots, if there are three roots, and for n roots if there are n roots. We then have

$$\frac{ds}{d\tau} = c_0 - \boldsymbol{\Theta}\cdot\mathbb{V}_a \tag{26}$$

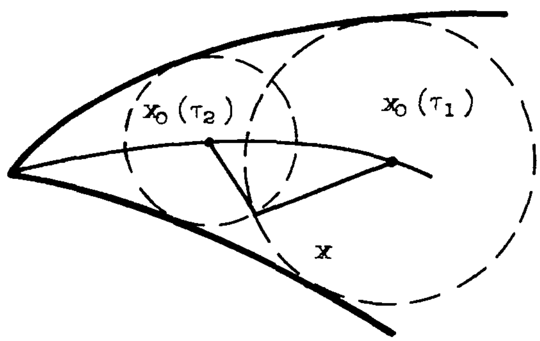
denoting by $\boldsymbol{\theta}$ the unit vector

$$\Theta = \frac{\mathbf{x} - \mathbf{x}_0}{|\mathbf{x} - \mathbf{x}_0|} \, . \tag{27}$$

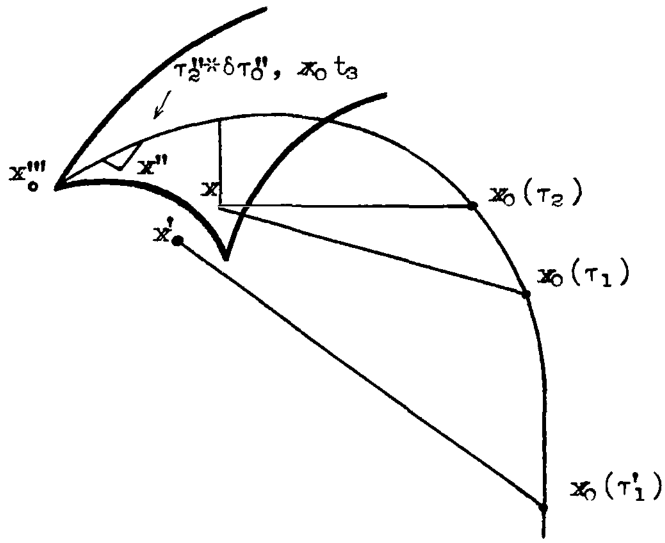
If $V_w < c_0$, we will always have $\frac{ds}{d\tau} = 0$. If the flight takes place in the subsonic range, the equation $\Gamma = 0$ cannot have more than one root in τ for fixed



$\mathbf{x}$, t, ξ. If the flight is supersonic, the quantity $\frac{ds}{d\tau}$ may change sign and two roots may be present. In the case of the accompanying diagram, we have two roots at the interior of the Mach conoid, namely, a zero root at the exterior



of the Mach conoid (zone of silence) and a double root on the Mach conoid itself. In $x_0(\tau_1)$, we have $\frac{ds}{d\tau} < 0$, whereas in $\mathbf{x}_0(\tau_2)$, we have $\frac{ds}{d\tau} > 0$. In more complex cases of supersonic flight, three and even more roots may be present, but this would necessitate an accelerated flight. We sketched a case /63
where this actually takes place, with the Mach conoid having a tilting edge. In $\mathbf{x}$, there are three roots; in $\mathbf{x}'$ there is only one root; in $\mathbf{x}''$ there are two roots; in $\mathbf{x}'''$ there is no root.



Let us return to eq.(23) and write $\Gamma(\underset{\sim}{P} - \mathfrak{M}) = \Gamma(t, x; \tau, \boldsymbol{\xi}), \boldsymbol{\xi} = (\xi, \eta, \zeta)$. Let us note that $\mathfrak{T}$ is written in the form of

$$\mathfrak{T} = \sum_{K} T_K(\tau)\, \mathfrak{T}_K(\boldsymbol{\xi}) \, , \tag{28}$$

where $T_1(\tau)$ is an (ordinary) function of only τ while $\mathfrak{T}_K(\xi, \eta, \zeta)$ is a distribution (function in ξ). Let us also denote by $t^* = T + \dfrac{|x - x_0(\tau) - \xi_0|}{c_0}$ the value of t which cancels Γ for fixed τ, $\boldsymbol{\xi}$, x; then, eq.(23) can be written as

$$< p_1(\mathfrak{M}), \phi(\mathfrak{M}) > = \int \sum_{K} T_K(\tau) < \mathfrak{T}_K, \iiint \frac{\phi(t^*, x)}{2\pi \Gamma_{t^*} c_0} \, dx > d\tau \, , \tag{29}$$

where < > denotes the explicit writing of a distribution over the variables ξ, η, ζ. A (non-one-to-one) correspondence exists between t and τ if x and $\boldsymbol{\xi}$ are fixed. We will attempt to make use of this fact in replacing the integration over τ by an integration over t. Let us first note that

$$\int d\tau < \, , > \; = \; < \, , \int d\tau > \, , \tag{30}$$

and that, consequently, we can write

$$< p_1(\mathfrak{M}), \phi(\mathfrak{M}) > = \sum_{K} < \mathfrak{T}_K, \iiiint \frac{T_K(\tau) \, \phi(t^*, x) \, d\tau \, dx}{2\pi \, \Gamma_t^* \, c_0} > . \tag{31}$$

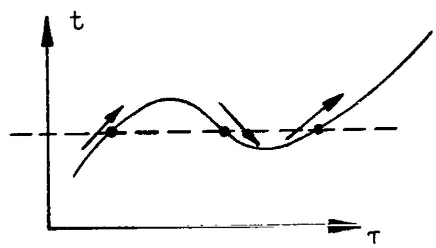
For changing the integration variable τ into the variable t, we must take the following relation into consideration: /64

$$\Gamma_{t^*}\, dt + \Gamma_{\tau}\, d\tau = 0, \tag{32}$$

which is valid at fixed x and ξ, such that we will have

$$\int_{-\infty}^{\infty} \frac{d\tau}{\Gamma_{t}^{*}} = \int_{-\infty}^{\infty} dt \sum_{i} \frac{1}{|\Gamma_{\tau_i}|}, \tag{33}$$

where the sum over i is extended to the roots in τ of the equation $\Gamma = 0$ at fixed t, x, ξ. In fact, if the axis of the quantities τ is traversed from $-\infty$ to $+\infty$, then the points indicated on the accompanying diagram are displaced in



the direction of the arrows; if $\Gamma_t > 0$, we have $\Gamma_\tau > 0$ if the arrow points toward increasing t while we have $\Gamma_\tau < 0$ if the arrow points toward decreasing t. Taking this statement into consideration, eq.(31) can be written in the form of

$$\langle p_1(\varkappa), \phi(\varkappa)\rangle = \sum_{K} \left\langle \mathcal{T}_K, \iiiint \sum_{i} \frac{T_K(\tau_i)\, \phi(t,x)}{2nc_0 |\Gamma_\tau(t,x;\tau_i,\xi)|}\, dt\, dx \right\rangle$$

$$= \iiiint \left\{ \sum_{K} \sum_{i} \left\langle \mathcal{T}_K, \frac{T_K(\tau_i)}{2n|\Gamma_\tau(t,x;\tau_i,\xi)|c_0} \right\rangle \right\} \phi(t,x)\, dt\, dx, \tag{34}$$

which demonstrates that we have

$$p_1(t,x) = \sum_{K} \sum_{i} \left\langle \mathcal{T}_K(\xi,\eta,\zeta), \frac{T_K(\tau_i)}{2n|\Gamma_\tau(t,x;\tau_i,\xi)|c_0} \right\rangle \tag{35}$$

where $\langle , \rangle$ denotes the explicit writing of $\mathcal{T}_K$ as distribution of the variables ξ,

η, ζ. Naturally, eq.(35) is valid only if { } in eq.(34) is a function, which we will prove; if this term were a distribution, we would have to return to the first line of eq.(34). It can be proved that the { } in question actually is a function, provided that $\frac{\partial^2 h}{\partial \xi^2}$, ... are also functions. To demonstrate this, it is sufficient to write - for example, for the fuselage $-\tau_k = \tau_k(\xi)\, \Sigma_k(\eta, \zeta)$ and /65

$$\left\langle \tau_K , \sum_i \frac{T_K}{|\Gamma_{\tau_i}|} \right\rangle = \left\langle \Sigma_K(\eta,\zeta) , \int_{-\infty}^{\infty} \mathcal{T}_K(\xi) \sum_i \frac{T_K(\tau_i)}{|\Gamma_\tau(t, x; \tau_i, \xi)|}\, d\xi \right\rangle. \tag{36}$$

It is obvious that

$$\mathcal{F}(\eta,\zeta) = \int_{-\infty}^{\infty} \mathcal{T}_K(\xi) \sum_i \frac{T_K[\tau_i(t, x; \xi, \eta, \zeta)]}{|\Gamma_\tau[t, x; \tau_i(t, x; \xi, \eta, \zeta), \xi, \eta, \zeta]|}\, d\xi \tag{37}$$

is a function, under the given hypotheses.

Theorem 17: Let, for a given event (t, x), the quantity $\tau_i(t, x; \xi, \eta, \zeta)$ be the roots in τ of the equation $\Gamma = 0$, considered as functions of ξ, η, ζ with

$$\Gamma = (t-\tau)^2 - |x - x_0(\tau) - \xi e_1(\tau) - \eta e_2(\tau) - \zeta e_3(\tau)|^2. \tag{38}$$

The acoustic pressure field, created by the aircraft schematized in theorem 16, is given by the formula

$$\begin{aligned} p_1(t, x) = {} & \frac{\rho_0}{2\pi c_0} \iint_{-\infty}^{\infty} \sum_i \left(\frac{V_A^2(\tau_i)}{|\Gamma_{\tau_i}|} \right)_{\zeta=0} \frac{\partial^2 h(\xi,\eta)}{\partial \xi^2}\, d\xi\, d\eta \, + \\ & + \frac{\rho_0}{2\pi c_0} \int_{-\infty}^{\infty} \sum_i \left(\frac{V_A^2(\tau_i)}{|\Gamma_{\tau_i}|} \right)_{\zeta=0} \frac{d^2 S(\xi)}{d\xi^2}\, d\xi \\ & - \frac{\rho_0}{2\pi c_0} \left\{ \frac{\partial}{\partial \zeta} \iint_{-\infty}^{\infty} \frac{\overline{\omega}_A(\xi, \eta, \tau_i(t, x; \xi, \eta, \zeta))}{|\Gamma_\tau(t, x, \tau_i(t, x; \xi, \eta, \zeta))|}\, d\xi\, d\eta \right\}_{\zeta=0} \end{aligned} \tag{39}$$

$$-\frac{\rho_0}{2\pi c_0}\left\{\nabla_{r}\cdot\int_{-\infty}^{\infty}\frac{\varpi_F\left(\xi,\tau_i(t,x;\xi,\eta,\zeta)\right)}{\left|\Gamma_\tau\left(t,x;\tau_i(t,x;\xi,\eta,\zeta)\right)\right|}\,d\xi\right\}_{\eta=\zeta=0}$$

In eq.(39), the integrands must be replaced by zero wherever Γ does not have a root in τ. /66

1.4.4 Remarks

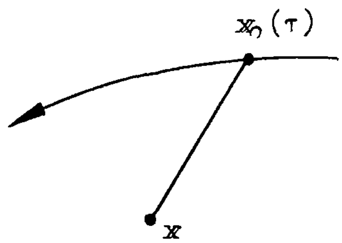
Let us consider the function

$$\Gamma(t,x;\tau) = (t-\tau)^2 - |x - x_0(\tau)|^2, \tag{40}$$

where the point $x_0(\tau)$ describes a given trajectory on variation of τ. Let

$$U_i(t,x) = \frac{1}{\Gamma_\tau\left(t,x;\tau_i(t,x)\right)} \tag{41}$$

where $\tau_i(t, x)$ is one root (of the roots) of $\Gamma = 0$ if this root (or roots) exist.



Theorem 18: Each of the functions $U_i(t, x)$, defined by eqs.(40) and (41), is a solution of the wave equation. If the motion of the point $x_0(\tau)$ is supersonic, then U_i becomes infinite in $d^{-\frac{1}{2}}$ if the point x approaches the Mach conoid with the vertex $x_0(t)$.

Let us omit the subscript i and demonstrate that we have

$$\frac{\partial^2 U}{\partial t^2} - c_0^2\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right) = 0 \ . \tag{42}$$

By expressing that Γ remains zero, we obtain

$$\Gamma_t \, dt + \Gamma_\tau \, d\tau + dx \cdot \Gamma_x = 0 \tag{43}$$

and, from this,

/67

$$\frac{\partial \tau}{\partial t} = -U \Gamma_t \quad , \quad \frac{\partial \tau}{\partial x} = -U \Gamma_x , \ldots , \tag{44}$$

whence

$$\begin{aligned}
\frac{\partial U}{\partial t} &= -U^2 \Gamma_{\tau t} + U^3 \Gamma_{\tau\tau} \Gamma_t \\
\frac{\partial^2 U}{\partial t^2} &= -2U \Gamma_{\tau t}\left(-U^2 \Gamma_{\tau t} + U^3 \Gamma_{\tau\tau} \Gamma_t\right) + 3U^2\left(-U^2 \Gamma_{\tau t} + U^3 \Gamma_{\tau\tau}\Gamma_t\right) \Gamma_{\tau\tau}\Gamma_t - \\
&\quad - U^2 \Gamma_{\tau tt} + U^3 \Gamma_{\tau\tau t} \Gamma_t + U^3 \Gamma_{\tau\tau} \Gamma_{tt} - U^4 \Gamma_{\tau\tau} \Gamma_{\tau t} \Gamma_t + \\
&\quad + U^3 \Gamma_{\tau\tau t} \Gamma_t - U^4 \Gamma_{\tau\tau\tau} (\Gamma_t)^2 \\
&= -U^2 \Gamma_{\tau tt} + U^3 \left\{ 2 (\Gamma_{\tau t})^2 + 2 \Gamma_{\tau\tau t} \Gamma_t + \Gamma_{\tau\tau} \Gamma_{tt} \right\} \\
&\quad - U^4 \left\{ 2 \Gamma_{\tau\tau} \Gamma_{\tau t} \Gamma_t + 4 \Gamma_{\tau\tau} \Gamma_{\tau t} \Gamma_t + \Gamma_{\tau\tau\tau} (\Gamma_t)^2 \right\} \\
&\quad + 3U^5 (\Gamma_{\tau\tau})^2 (\Gamma_t)^2 \\
&= -U^2 \frac{\partial}{\partial \tau} (\Gamma_{tt}) + U^3 \left\{ \frac{d^2}{\partial \tau^2} \left[(\Gamma_t)^2\right] + \Gamma_{\tau\tau} \Gamma_{tt} \right\} - \\
&\quad - U^4 \left\{ 3 \Gamma_{\tau\tau} \frac{\partial}{\partial \tau} \left[(\Gamma_t)^2\right] + \Gamma_{\tau\tau\tau} (\Gamma_t)^2 \right\} + 3U^5 (\Gamma_{\tau\tau})^2 (\Gamma_t)^2 .
\end{aligned} \tag{45}$$

The derivatives $\frac{\partial^2 U}{\partial x^2}$, ... are given by analogous formulas, so that it is necessary to pose

$$\left\{ \begin{aligned} &\Gamma_{tt} - c_0^2 \left(\Gamma_{xx} + \Gamma_{yy} + \Gamma_{zz} \right) = A , \\ &(\Gamma_t)^2 - c_0^2 \left\{ (\Gamma_x)^2 + (\Gamma_y)^2 + (\Gamma_z)^2 \right\} = B , \end{aligned} \right. \tag{46}$$

which results in

$$\frac{\partial^2 U}{\partial t^2} - c_0^2\left(\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2}\right) = -U^2 \frac{\partial A}{\partial \tau} + U^3\left(\frac{\partial^2 B}{\partial \tau^2} + \Gamma_{\tau\tau} A\right) - $$
$$ - U^4\left(3\Gamma_{\tau\tau}\frac{\partial B}{\partial \tau} + \Gamma_{\tau\tau\tau} B\right) + 3U^5(\Gamma_{\tau\tau})^2 B \ . \qquad (47)$$

However, it is obvious that we obtain

$$A = 8 \quad , \quad B = 4\Gamma \qquad (48)$$

so that /68

$$\frac{\partial A}{\partial \tau} = 0 \quad , \quad \frac{\partial^2 B}{\partial \tau^2} + A\,\Gamma_{\tau\tau} = 12\,\Gamma_{\tau\tau} , \qquad (49)$$

from which it follows that

$$\frac{\partial^2 U}{\partial t^2} - c_0^2 \Delta U = 4\Gamma\,\frac{(\Gamma_{\tau\tau})^2 - \Gamma_{\tau\tau\tau}\Gamma_\tau}{(\Gamma_\tau)^5} = 0, \qquad (50)$$

since it is necessary to set $\Gamma = 0$.

This calculation is rather inelegant and it is quite obvious that one can do better. It is known that, for fixed τ, the quantity $\delta(\Gamma)$ is a solution so that, with arbitrary $f(\tau)$,

$$< \delta(\Gamma), f(\tau) > = < \sum_i \frac{\delta(\tau - \tau_i)}{|\Gamma_{\tau_i}|}, f(\tau) > = \sum_i \frac{f(\tau_i)}{|\Gamma_{\tau_i}|} , \qquad (51)$$

will be a solution, CQFD (!).

Let us now proceed to the last part of the theorem, and let x^* be a point on the Mach conoid at the instant t, so that

$$\Gamma_\tau(t, x^*; \tau^*) = \Gamma(t, x^*; \tau^*) = 0 \qquad (52)$$

denoting by τ^* the root n, in τ, of $\Gamma = 0$ which corresponds to x^*. If the vicinity of a tilting edge of the Mach conoid is excluded, it follows that

$$\Gamma_{\tau^*\tau^*} \neq 0, \quad |\Gamma_{x^*\tau^*}| \neq 0, \tag{53}$$

and, consequently, if x is close to x^*,

$$\begin{cases} \Gamma \cong (x - x^*) \cdot \Gamma_{x^*} + \frac{1}{2}(\tau - \tau^*)^2 \, \Gamma_{\tau^*\tau^*} + (\tau - \tau^*)(x - x^*) \cdot \Gamma_{\tau^* x^*} \\ \Gamma_\tau \cong (\tau - \tau^*) \, \Gamma_{\tau^*\tau^*} + (x - x^*) \cdot \Gamma_{x^*} + \dots \end{cases} \tag{54}$$

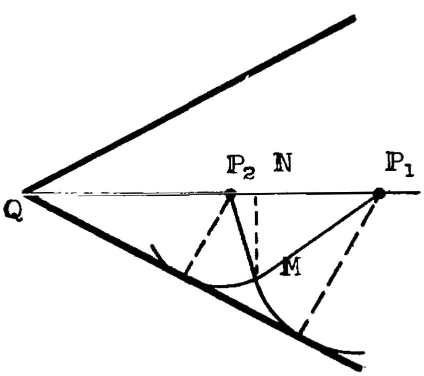
retaining only the dominant terms. The first equation shows that no root in τ exists if $(x - x^*) \cdot \Gamma_{x^*}\Gamma_{\tau^*\tau^*} > 0$; if the inverse inequality occurs, we will /69
have two roots

$$\tau_1 - \tau^* \cong \tau^* - \tau_2 \cong \sqrt{\frac{-2\,(x - x^*) \cdot \Gamma_{x^*}}{\Gamma_{\tau^*\tau^*}}} \tag{55}$$

resulting in

$$\Gamma_{\tau_1} \cong -\Gamma_{\tau_2} \cong \sqrt{-2\,(x - x^*) \cdot \Gamma_{x^*} \, \Gamma_{\tau^*\tau^*}}, \tag{56}$$

which proves the theorem.



Let us now assume that the motion of $x_0(\tau) = \mathbb{P}(\tau)$ is rectilinear, uniform, and supersonic, yielding

$$\begin{cases} \Gamma_{\tau\tau} = -\frac{1}{2} \cdot \beta^2 = \text{Const} < 0 \\ \beta^2 = \frac{V_0^2}{c_0^2} - 1 \qquad V_0 = \frac{dx_0}{d\tau} \end{cases} \tag{57}$$

so that

$$\Gamma \equiv -\beta^2(\tau-\tau_1)(\tau-\tau_2) \tag{58}$$

if τ_1 and τ_2 are the two roots in τ of $\Gamma = 0$, resulting in

$$\Gamma_{\tau_1} = -\Gamma_{\tau_2} = \beta^2(\tau_2-\tau_1)\,. \tag{59}$$

Let us now project $\mathbb{M}$ or $\mathbb{N}$ onto the axis of the Mach cone and let us pose

$$|O\mathbb{N}| = x \qquad |\mathbb{N}\mathbb{M}| = r\,, \quad |O\mathbb{P}| = \xi \tag{60}$$

yielding

$$\beta^2\xi^2 - 2M_0^2\,x\,\xi + M_0^2(x^2+r^2) = 0 \tag{61}$$

by writing $|\mathbb{M}\,\mathbb{P}| = c_0|\mathbb{O}\,\mathbb{P}|$, so that

$$\xi_2-\xi_1 = 2\frac{M_0}{\beta^2}(x^2-\beta^2r^2)^{\frac{1}{2}} = c_0(\tau_2-\tau_1)\,. \tag{62}$$

Thus, in this case,

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$$U_1 = -U_2 = \frac{c_0}{2\sqrt{x^2-\beta^2r^2}}\,. \tag{63}$$

Theorem 18: Let us consider an aircraft in rectilinear and uniform supersonic cruising flight; with aircraft-fixed axes, the pressure field is given by

$$\begin{aligned} p_1(\xi,\eta,\zeta) = {} & \frac{\rho_0V_0^2}{2\pi}\int\frac{S''(\xi_1)}{\sqrt{(\xi-\xi_1)^2-\beta^2(\eta^2+\zeta^2)}}\,d\xi_1 \\ & + \frac{\rho_0V_0^2}{2\pi}\iint\frac{h''_{\xi\xi}(\xi_1,\eta_1)}{\sqrt{(\xi-\xi_1)^2-\beta^2[(\eta-\eta_1)^2+\zeta^2]}}\,d\xi_1\,d\eta_1 \\ & + \frac{\rho_0}{2\pi}\frac{\partial}{\partial\zeta}\iint\frac{\overline{\omega}_A(\xi_1,\eta_1)}{\sqrt{(\xi-\xi_1)^2-\beta^2[(\eta-\eta_1)^2+\zeta^2]}}\,d\xi_1\,d\eta_1 \end{aligned} \tag{64}$$

$$+\frac{\rho_0}{2\pi}\nabla\cdot\int\frac{\overline{\omega}_F(\xi_1)}{\sqrt{(\xi-\xi_1)^2-\beta^2[\eta^2+\zeta^2]}}\,d\xi_1$$

1.4.5 Asymptotic Behavior

It is necessary to differentiate between the distal behavior and the proximal behavior, as we had explained at the end of Section 1.2.3. We leave to the reader the task of proving the following theorem.

Theorem 18a: The distal asymptotic behavior (far from the aircraft and far from the Mach cone) is given by the following formula for the case of stationary flight:

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$$p_1 \cong \frac{1}{2\pi}P_w\frac{\partial}{\partial\zeta}\frac{1}{\sqrt{\xi^2-\beta^2(\eta^2+\zeta^2)}}+\frac{1}{2\pi}\mathbb{P}_F\cdot\nabla\frac{1}{\sqrt{\xi^2-\beta^2(\eta^2+\zeta^2)}}+\cdots \tag{65}$$

where P_w and $\mathbb{P}_F$ denote, respectively, the lift force and the vector of the transverse force exerted by the air on the wing and on the fuselage. The volume effects furnish a weaker contribution; if these effects are to be taken into consideration, the distal expansion must be continued further, and the following must be written:

$$\begin{aligned}p_1 \cong {} & \frac{1}{2\pi}P_w\frac{\partial}{\partial\zeta}\frac{1}{\sqrt{\xi^2-\beta^2(\eta^2+\zeta^2)}}+\frac{1}{2\pi}\mathbb{P}_F\cdot\nabla\frac{1}{\sqrt{\xi^2-\beta^2(\eta^2+\zeta^2)}}\\ & +\frac{1}{2\pi}\mathfrak{M}_w\frac{\partial}{\partial\zeta}\frac{\xi}{\{\xi^2-\beta^2(\eta^2+\zeta^2)\}^{3/2}}-\frac{1}{2\pi}e_1\cdot\left\{\mathfrak{M}_F\wedge\nabla\frac{\xi}{\{\xi^2-\beta^2[\eta^2+\zeta^2]\}^{3/2}}\right\}\\ & +\frac{1}{2\pi}(\mathfrak{V}_w+\mathfrak{V}_F)\frac{2\xi^2+\beta^2(\eta^2+\zeta^2)}{\{\xi^2-\beta^2(\eta^2+\zeta^2)\}^{5/2}}+\cdots\end{aligned} \tag{66}$$

denoting by $\mathfrak{M}_w$ the moment resulting in $\xi=\eta=\zeta=0$ from the forces exerted on the wing ($\mathfrak{M}_w = \iint p_0\overline{\omega}\,\xi\,d\xi\,d\eta$) and denoting by $\mathfrak{M}_F$ the moment vector resulting in O from the forces exerted on the fuselage, where $\mathfrak{V}_w$ and $\mathfrak{V}_F$ are the volumes, respectively, of the wing and the fuselage.

Let us now return to the cruising flight and consider the proximal behavior, i.e., the behavior in the vicinity of the Mach cone. This makes us pose

$$\xi = \xi_0 + \beta r \qquad \eta = r\cos\theta \qquad \zeta = r\sin\theta \tag{67}$$

by means of which we obtain /72

$$(\xi-\xi_1)^2 - \beta^2\left[(\eta-\eta_1)^2 + (\zeta-\zeta_1)^2\right] \cong 2\beta r\left\{(\xi_0-\xi_1) + \beta\left[\eta_1\cos\theta + \zeta_1\sin\theta\right]\right\} \tag{68}$$

so that we will have

$$\begin{aligned} p_1 \cong\ & \frac{\rho_0 V_0^2}{2\pi\sqrt{2\beta r}}\int \frac{S''_{\xi\xi}(\xi_1)}{\sqrt{\xi_0-\xi_1}}\,d\xi_1 + \frac{\rho_0 V_0^2}{2\pi\sqrt{2\beta r}}\iint \frac{h''_{\xi\xi}(\xi_1,\eta_1)}{\sqrt{\xi_0-\xi_1+\beta\eta_1\cos\theta}}\,d\xi_1\,d\eta_1 \\ & - \frac{\rho_0}{2\pi\sqrt{2\beta r}}\left\{\frac{\partial}{\partial\zeta_1}\iint \frac{\varpi_A(\xi_1,\eta_1)\quad d\xi_1\,d\eta_1}{\sqrt{\xi_0-\xi_1+\beta[\eta_1\cos\theta+\zeta_1\sin\theta]}}\right\}_{\zeta_1=0} \\ & - \frac{\rho_0}{2\pi\sqrt{2\beta r}}\left\{\nabla\cdot\int \frac{\varpi_F(\xi_1)\quad d\xi_1}{\sqrt{\xi_0-\xi_1+\beta[\eta_1\cos\theta+\zeta_1\sin\theta]}}\right\}_{\eta_1=\zeta_1=0}. \end{aligned} \tag{69}$$

Let us pose

$$\xi^* = \xi_1 - \beta(\eta_1\cos\theta + \zeta_1\sin\theta) \tag{70}$$

and then effect the change in variables

$$(\xi_1,\eta_1,\zeta_1) \Longrightarrow (\xi^*,\eta_1,\zeta_1) \tag{71}$$

whose Jacobian is equal to unity; this will yield

$$\begin{aligned} p_1 \cong \frac{\rho_0}{2\pi\sqrt{2\beta r}}\int \frac{d\xi^*}{\sqrt{\xi_0-\xi^*}}&\left\{V_0^2 S''_{\xi^*\xi^*}(\xi^*) + V_0^2\int h''_{\xi\xi}(\xi^*+\beta\eta_1\cos\theta,\eta_1)\,d\eta_1\right. \\ & - \left(\frac{\partial}{\partial\zeta_1}\int \varpi_A(\xi^*+\beta[\eta_1\cos\theta+\zeta_1\sin\theta],\eta_1)\ d\eta_1\right)_{\zeta_1=0} - \end{aligned} \tag{72}$$

$$-\left(\nabla_{\tau} \cdot \varpi_F \left(\xi^* + \beta[\eta_1 \cos\theta + \zeta_1 \sin\theta]\right)\right)_{\eta_1=\zeta_1=0}$$

If we pass from moving axes to fixed axes, the right-hand side of eq.(67) will correspond to a sound beam whose direction is defined by the unit vector

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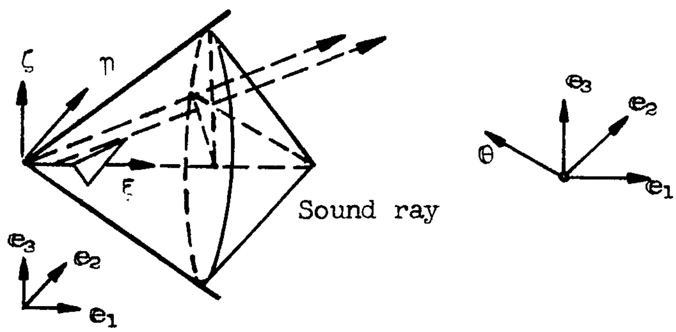
$$\Theta = e_1 M_0^{-1} + \sqrt{1 - M_0^{-2}}\,\left(e_2 \cos\theta + e_3 \sin\theta\right) \tag{73}$$

leading to the concept of:

Equivalent Fuselage.

This is a fuselage depending on the direction Θ whose law of areas is



$S_e(\xi, \Theta)$, defined by

$$\begin{aligned}\frac{\partial S_e(\xi,\Theta)}{\partial \xi} = {} & \frac{\partial S(\xi)}{\partial \xi} + \int \frac{\partial h(\xi + M_0 e_2\cdot\Theta\,\eta, \eta)}{\partial \xi}\,d\eta \\ & - M_0\, e_3\cdot\Theta \int V_0^{-2}\,\varpi_A\left(\xi + M_0 e_2\cdot\Theta\,\eta, \eta\right) d\eta \\ & - M_0\,\Theta \cdot \int V_0^{-2}\,\varpi_F\left(\xi + M_0 e_2\cdot\Theta\,\eta, \eta\right) d\eta\end{aligned} \tag{74}$$

so that eq.(72) is reduced to

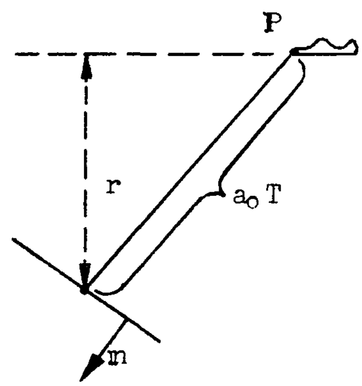
$$p_1 \sim \frac{\rho_0 V_0^2}{2\pi\sqrt{2\beta r}}\,\frac{\partial^2}{\partial \xi_0^2} \int \frac{S_e(\xi,\Theta)}{\sqrt{\xi_0 - \xi}}\,d\xi \tag{75}$$

exactly as if the fuselage of revolution with the law of areas $S_e(\xi;\Theta)$ were isolated.

Let $\mathbb{P}(\tau)$ be the position (dotted line) of the aircraft at the instant τ, representing the position relative to a non-moving reference point! At the instant $t = \tau + T$, the point whose position, in the case of moving axes, is located in eq.(67), will actually be located in /74

$$\mathbb{M} = \mathbb{P}(\tau) + a_0 T\, \Theta + \xi_0 e_1 , \tag{74}$$

and, since T is large, everything proceeds as though the pressure p, given by



eq.(75), were applied in

$$\begin{aligned} \mathbb{M} &= \mathbb{P}(\tau) + a_0 T\, \Theta + x\, n \\ x &= -\xi_0 M_0 \end{aligned} \tag{75}$$

yielding

$$\beta r = \frac{M^2-1}{M}\, a_0 T. \tag{76}$$

Theorem 19: The proximal asymptotic behavior (far from the aircraft, near the Mach wave) of the acoustic field created by an aircraft in supersonic cruising flight is given by the formulas

$$\left\{ \begin{aligned} & p_1(T, x; \tau, \Theta) = 2^{-1/2} \rho_0 V_0^2 M_0^{1/2} (M_0^2-1)^{-1/2} (a_0 T)^{-1/2} G(x, \Theta) && a) \\ & t = \tau + T && b) \\ & \mathbb{M} = \mathbb{P}(\tau) + a_0 T\, \Theta + n\, x \qquad (n = \Theta) && c) \end{aligned} \right. \tag{77}$$

where $\mathbb{P}(\tau)$ is the position of the aircraft, assumed as represented by the broken line, at the instant τ; $\mathbf{n}$ is the unit vector normal to the Mach wave which, at the instant $t = \tau + T$, passes to the point $\mathbb{M}_0 = \mathbb{P}(\tau) + a_0 T\, \Theta$ where Θ is the unit vector tangent in $\mathbb{P}(\tau)$ to the rectilinear sound ray going from $\mathbb{P}$ to $\mathbb{M}_0$ (it will be recalled that $\Theta \cdot \mathbf{e}_1 = -\mu_0^{-1}$); finally, the function $G(x)$ is connected with /75 the Whitham function

$$F(\xi_0, \Theta) = \frac{1}{2\pi} \frac{\partial^2}{\partial \xi_0^2} \int \frac{S_e(\xi, \Theta)}{\sqrt{\xi_0 - \xi}}\, d\xi, \tag{78}$$

by the formula

$$G(x, \Theta) = F\left(-\frac{x}{\mu_0}, \Theta\right). \tag{79}$$

Let us return to the case in which the flight is varied. In that case, we must replace $\sqrt{(\xi - \xi_1)^2 - \beta^2[(\eta - \eta_1)^2 + (\zeta - \zeta_1)^2]}$ by $\frac{c_0}{2|\Gamma_\tau|}$, with

$$\Gamma = (t - \tau)^2 - a_0^{-2}\, |\mathbb{Q}\mathbb{M}|^2 \tag{80}$$

where $(\mathbb{M}, t)$ is an arbitrary instant point, while

$$\mathbb{Q} = \mathbb{P}(\tau_0) + \xi_1 \mathbf{e}_1(\tau_0) + \eta_1 \mathbf{e}_2(\tau_0) + \zeta_1 \mathbf{e}_3(\tau_0). \tag{81}$$

Since we wish to study the proximal asymptotic behavior, we must substitute eq.(77c) and make $T \to \infty$, yielding

$$\Gamma = (T + \tau_0 - \tau)^2 - a_0^{-2} \left|(a_0 T + x)\Theta(\tau_0) + \xi_1 \mathbf{e}_1(\tau_0) + \eta_1 \mathbf{e}_2(\tau_0) + \zeta_1 \mathbf{e}_3(\tau_0) + \mathbb{P}(\tau) - \mathbb{P}(\tau_0)\right|^2 \tag{82}$$

In accordance with the construction procedure, we have

$$\Gamma = \Gamma_\tau = 0 \tag{83}$$

for

$$\tau = \tau_0, \quad \mu_0 x + \xi_1 = 0; \quad \eta_1 = \zeta_1 = 0 \tag{84}$$

so that, in the vicinity of these values, we can write

$$\Gamma \cong \varepsilon\, \mathcal{U}(T; \tau_0, \Theta)\, (\tau - \tau_0)^2 - 2\, \frac{T(\xi^* + \mu_0 x)}{a_0 \mu_0} \tag{85}$$

with

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$$\xi^{x} = \xi_1 - \mu_0[e_2 \cdot \Theta\, \eta_1 + e_3\, \Theta\, \eta_2] \tag{86}$$

$$\left\{ \begin{array}{l} \varepsilon\, \mathcal{U} = \frac{1}{2}\Gamma_{\tau\tau}\left(\tau = \tau_0,\ \mu_0 x + \xi_1 = 0,\ \eta_1 = 0, \zeta_1 = 0\right) \\ \varepsilon = \pm 1 \qquad \mathcal{U} > 0 \end{array} \right. \tag{87}$$

The roots τ_i of $\Gamma = 0$ thus will be

$$\tau_i = \tau_0 \pm \sqrt{\frac{2T\,(\xi^{*} + \mu_0 x)\,\varepsilon}{a_0 \mu_0\, \mathcal{U}}}, \tag{88}$$

resulting in

$$|\Gamma_{\tau_i}| = \sqrt{\frac{\varepsilon\, \mathcal{U}\, T\, (\xi^{*} + \mu_0 x)}{a_0 \mu_0}}. \tag{89}$$

Theorem 20: Let V_a be the aircraft speed and let γ_a be its acceleration vector; then, we pose

$$\mathcal{U}(T; \tau, \Theta) = \varepsilon\left\{\frac{V_a^2(\tau)}{a_0^2} - 1 - \frac{\gamma(\tau)\cdot\Theta(\tau)}{a_0}\, T\right\} \tag{90}$$

with $\varepsilon = \pm 1$, so that $\mathcal{U} > 0$. We exclude the vicinity of

$$T^{*} = \frac{V_a^2 - a_0^2}{a_0\, \gamma \cdot \Theta}, \tag{91}$$

and assume that, during the interval of time

$$\Delta\tau = 2\sqrt{\frac{2T\ell}{a_0 \mu_0 \mathcal{U}}} \tag{92}$$

the instantaneous equivalent fuselage $S_e(\Sigma; \theta, \tau)$ does not undergo appreciable variations so that eq.(77) is valid under the condition of replacing $(\mu_0^2 - 1)a_0 T$ by $a_0 T\mathcal{U}\,(T;\tau,\Theta)$ and replacing $G(x, \Theta)$ by /77

$$G(x,\Theta,\tau) = F_{\varepsilon}\left(-\frac{x}{\mu(\tau)}, \Theta(\tau), \tau\right) \tag{93}$$

with

$$\begin{cases} F_+(\xi,\Theta,\tau) = \dfrac{1}{2\pi}\dfrac{\partial^2}{\partial\xi^2}\displaystyle\int_{-\infty}^{\xi} \dfrac{S_e(\xi_1,\Theta,\tau)}{\sqrt{\xi-\xi_1}}\, d\xi_1 \\ F_-(\xi,\Theta,\tau) = \dfrac{1}{2\pi}\dfrac{\partial^2}{\partial\xi_0^2}\displaystyle\int_{\xi}^{+\infty} \dfrac{S_e(\xi_1,\Theta,\tau)}{\sqrt{\xi_1-\xi}}\, d\xi_1 \end{cases} \tag{94}$$

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CHAPTER II /II,1

GENERAL PRINCIPLES OF OSCILLATIONS

2.1 Establishing an Oscillatory Regime

2.1.1 Preliminaries

In this Chapter, we will investigate the solutions of the wave equation, written in the form of

$$\psi = e^{i\omega t}\, u(\mathbf{x}) \tag{1}$$

with ω being real. The function u proves the Helmholtz equation

$$\Delta u + k^2 u = 0 \qquad k = \frac{\omega}{c} \tag{2}$$

where the pulsation is denoted by ω, the frequency by $\omega/2n = \frac{kc}{2n}$, the wave number by k, and the wavelength by k^{-1}.

Among the simple solutions of eq.(2), the spherical or cylindrical plane waves are of greatest interest. A solution for plane waves has the form of $u = f(\alpha x + \beta y + \gamma z)$ with $\alpha^2 + \beta^2 + \gamma^2 = 1$, from which $f'' + k^2 f = 0$ is obtained in such a manner that the following expression results for a plane wave:

$$u = A \exp\{i\,\mathbf{k}\cdot\mathbf{x}\} \qquad \mathbf{k} = (k\alpha, k\beta, k\gamma) = k\boldsymbol{\omega}. \tag{3}$$

By re-establishing the dependence relative to time

$$\phi = A\, e^{i\omega\left(t - \frac{\boldsymbol{\omega}\cdot\mathbf{x}}{c}\right)} \tag{4}$$

we obtain a solution of the wave equation by plane waves. A solution of eq.(2) for spherical waves is obtained by taking $u = f(r)$ where $r = |\mathbf{x}| = \{x^2 + y^2 + z^2\}^{1/2}$; the quantity f must prove the equation $f'' + 1/r\, f + k^2 f = 0$ which reduces to $F'' + k^2 F = 0$ by assuming $rf = F$ so that a spherical Helmholtz wave will assume the following form: /II,2

$$u = A\,\frac{e^{\pm ikr}}{r}, \tag{5}$$

i.e., in re-introducing the time factor

$$\phi = A \frac{e^{i\omega(t \pm r/c)}}{r}, \tag{6}$$

such that the plus sign corresponds to an antiradiant wave while the minus sign corresponds to a radiant wave. By superposition, the following two solutions are obtained:

$$u = \frac{\sin(kr)}{r} \text{ (i)}, \quad u = \frac{\cos(kr)}{r} \text{ (ii)}. \tag{7}$$

The solution (i) is everywhere regular and vanishes at infinity; it is also a solution which is regular in $r < \frac{(2n+1)^n}{k}$ where n is an integer, and vanishes for $r = \frac{(2n+1)^n}{k}$; thus, it must be expected that the uniqueness theory for the Helmholtz equation exhibits a few peculiar characteristics. It should also be noted that $\frac{\cos(kr)}{r}$ is a singular solution at the origin and that the behavior, in the vicinity of $r = 0$, is in $1/r$, i.e., the same as for the Laplace equation. The reason for this is quite simple: If u is singular at the origin, its derivatives are quite superior to the function itself and $|\Delta u| \gg k^2 u$. For cylindrical waves, we have $u = f(r)$ with $r = \sqrt{x^2 + y^2}$, while f proves the Bessel equation of zero order:

$$f'' + \frac{1}{r} f' + k^2 f = 0 \tag{8}$$

of which two independent solutions are known, playing the respective roles of eq.(7), namely,

$$u = J_0(kr) \text{ (i)}, \qquad u = Y_0(kr) \text{ (ii)}. \tag{9}$$

The analogous solutions for spherical waves $\frac{e^{\pm ikr}}{r}$ are as follows: /II.3

$$\begin{cases} H_0^{(1)}(kr) = J_0(kr) + iY_0(kr), \\ H_0^{2}(kr) = J_0(kr) - iY_0(kr). \end{cases} \tag{10}$$

It can be demonstrated that, for large values of r, we have

$$H_0^{(1,2)}(kr) \sim \frac{e^{\pm ik(r-\pi/4)}}{\sqrt{2\pi r}} \qquad \begin{matrix} + \Longleftrightarrow 1 \\ - \Longleftrightarrow 2 \end{matrix} \tag{11}$$

making it obvious that $e^{i\omega t}H_0^{(1)}(kr)$ is an antiradiant cylindrical wave in the sense of Chapter I, while $e^{i\omega t}H_0^{(2)}(kr)$ is a radiant cylindrical wave.

2.1.2 Study of Cauchy's Problem

To understand the mode of generation of Helmholtz waves, let us consider the Cauchy problem in accordance with

$$\begin{cases} \dfrac{\partial^2\phi}{\partial t^2} - \Delta\phi = f(x)\, e^{i\omega t}, & (c=1) \\ \phi(t,x) = g(x), \\ \dfrac{\partial\phi(t,x)}{\partial t} = h(x), \end{cases} \tag{12}$$

whose solution is known from the theorem 11 of Chapter I, namely,

$$\phi(t,x) = \int_{\tau}^{t} \tau\, \mathfrak{M}_\tau^*(f)\, e^{i\omega(t-\tau)}\, d\tau + t\, \mathfrak{M}_t^*(g) + \frac{\partial}{\partial t}\left\{t\, \mathfrak{M}_t^*(h)\right\}, \tag{13}$$

which is unique, as we know. Let us suppose that f, g, h are regular, bounded, and zero outside of a sphere whereas, for each finite $|x|$, the quantities $\mathfrak{M}_t^*(f)$, $\mathfrak{M}_t^*(g)$, $\mathfrak{M}_t^*(h)$ vanish identically in the neighborhood of infinity in t. From this, it can obviously be deduced that

/II,4

$$\lim_{t\to\infty} \phi(t,x) = e^{i\omega t}\, u(x), \tag{14}$$

with

$$u(x) = \int_0^\infty \tau\, \mathfrak{M}_\tau^*(f)\, e^{-i\omega\tau}\, d\tau. \tag{15}$$

In accordance with the definition of the operation of the mean, we have

$$u(x) = \frac{1}{4\pi}\iiint \frac{f(x-\xi, y-\eta, z-\zeta)}{\sqrt{\xi^2+\eta^2+\zeta^2}}\, e^{-ik(\xi^2+\eta^2+\zeta^2)^{\frac{1}{2}}}\, d\xi\, d\eta\, d\zeta \tag{16}$$

$$= f * \frac{e^{-ikr}}{4\pi r} \qquad \text{(convolution)}$$

and u proves that

$$\Delta u + k^2 u = -f. \tag{17}$$

Thus, for very large t, the effect of the initial conditions becomes established and only a harmonic vibration remains, formed by the second term of eq.(12a).

It should be noted that $f * \frac{e^{+ikr}}{4\pi r}$ is also a solution of eq.(17) but that this solution is not a permanent solution for the original problem. It can be guessed already that, in interpreting the solutions of the Helmholtz equation, the radiant waves obviously play a preferential role.

If $|x| \gg R_0$ where R_0 is the radius of the sphere outside of which f vanishes, then the solution (16) behaves approximately like a radiant spherical wave, i.e.,

$$u \sim \left\{\frac{1}{4\pi}\iiint f(x)\, dx\right\} \frac{e^{-ik|x|}}{|x|}. \tag{18}$$

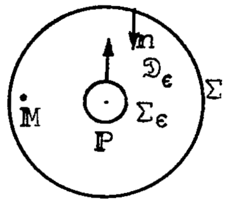
It should be emphasized that the uniqueness of the solution of a problem relative to the Helmholtz equation can be obtained only by imposing a condition as to the radiant or antiradiant character at infinity. Sommerfeld's radiation condition is a condition of this type. /II,5

2.2 Green's Formula. Conservation of Energy

If the functions u and v are provided with continuous second derivatives in a three-dimensional domain $\mathfrak{D}$, limited by a regular surface Σ with continuity of the derivatives up to Σ, we will have

$$\iiint_{\mathfrak{D}} (u\Delta v - v\Delta u)\, dx = \iint_{\Sigma} \left(v\frac{du}{dn} - u\frac{dv}{dn}\right) dS, \tag{1}$$

where the normal derivative $\frac{d}{dn}$ is evaluated in a direction pointing toward the interior of $\mathfrak{D}$. Let us apply this formula by using, for u, a regular solution of the Helmholtz equation and by using, for v, a spherical wave $v = \frac{e^{\pm ikr}}{4\pi r}$ where r denotes ($r = |x - \xi|$) the distance of a moving point $\mathbb{P} = \xi$ of $\mathfrak{D}$ from a



fixed point $\mathbb{P} = x$. If $\mathbb{P}$ is located within $\mathfrak{D}$, let us take the precaution of adding, to Σ, a small sphere with the center $\mathbb{P}$ and the radius ϵ, and let us replace $\mathfrak{D}$ by $\mathfrak{D}_\epsilon$, comprised between Σ and Σ_ϵ. It is obvious that the volume integral will vanish as soon as the integral extended to Σ_ϵ tends toward $u(\mathbb{P}) = u(x)$ when $\epsilon \to 0$.

Theorem 1: Let $\mathfrak{D}$ be a bounded domain, limited by an exterior surface and by one or several interior surfaces whose entity is denoted by Σ since these surfaces are regular, i.e., are provided with a tangent plane which is continuously dependent on the position. Let $u(x)$ be a regular solution of $\Delta u + k^2 u = 0$, i.e., be twice continuously differentiable at the interior of $\mathfrak{D}$ and once continuously differentiable up to Σ, yielding /II,6

$$0 = \frac{1}{4\pi} \iint_{\Sigma} \left\{ \frac{e^{\pm ik|x-\xi|}}{|x-\xi|} \frac{du(\xi)}{dn_\xi} - u(\xi) \frac{d}{dn_\xi} \frac{e^{\pm ik|x-\xi|}}{|x-\xi|} \right\} dS_\xi + 1(\mathfrak{D})\, u(x) \tag{2}$$

with

$$1(\mathfrak{D}) = \begin{cases} 1, & \text{if } x \text{ is within } \mathfrak{D}, \\ 0, & \text{if } x \text{ is outside of } \mathfrak{D}. \end{cases} \tag{3}$$

The normal derivative $\frac{d}{dn}$ is evaluated in the direction pointing toward the interior of $\mathfrak{D}$. In the case of two-dimensional space, eq.(2) is valid under the condition that $\frac{1}{4\pi} \frac{e^{\pm ik|x - \xi|}}{|x - \xi|}$ is replaced by $\frac{1}{4i} H^{(1,2)}(k|x - \xi|)$.

Problem 1: Derive from the preceding theorem that, at three (respectively, two) dimensions, $\frac{e^{\pm ikr}}{4\pi r}$ $\left(\text{resp. } \frac{1}{4i} H_0^{(1,2)}(kr)\right)$ is the solution of

$$\Delta u + K^2 u = -\delta, \tag{4}$$

where δ is the unit Dirac mass, placed at the origin $x = 0$.

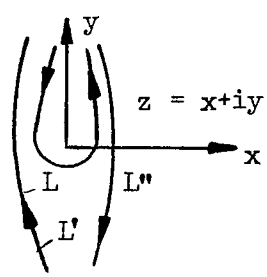
Problem 2: Establish the relation

$$\frac{1}{4\pi}\int_{-\infty}^{\infty} \frac{e^{\pm iK\sqrt{r^2+z^2}}}{\sqrt{r^2+z^2}}\, dz = \frac{1}{2\pi}\int_0^{\infty} \cos(Kr\coth\varphi)\, d\varphi \pm \frac{i}{2\pi}\int_0^{\infty} \sin(Kr\coth\varphi)\, d\varphi \tag{5}$$

Problem 3: It will be recalled that

$$J_\nu(a) = \frac{1}{2\pi i}\int_L z^{-\nu-1} e^{\frac{a}{2}\left(z-\frac{1}{z}\right)}\, dz \qquad a>0 \tag{6}$$

where L is a contour which starts from $+i\infty$, encircles the origin, and goes to



$+i\infty$. It will also be recalled that

/II,7

$$Y_\nu(a) = \frac{J_\nu(a)\cos(\pi\nu) - J_{-\nu}(a)}{\sin(\pi\nu)}. \tag{7}$$

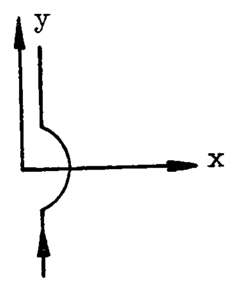
In eq.(6), demonstrate that the result will not change on replacing L by L + L', where L' goes from $+i\infty$ to $-i\infty$ without passing through the origin. By conviently selecting L and L', demonstrate that

$$J_\nu(a) = \frac{1}{2\pi i}\int_\Gamma z^{-\nu-1} e^{\frac{a}{2}\left(z-1/z\right)}\, dz \tag{8}$$

where Γ is the contour of the accompanying diagram, with a half-circle centered at the origin of vanishing radius. Perform the explicit calculation of eq.(8), by posing $z = e^{\varphi+i\theta}$ and by deriving from this:

$$\left\{ \begin{aligned} J_0(a) &= \frac{2}{\pi}\int_0^{\infty} \sin\left\{a \coth\varphi\right\} d\varphi \\ Y_0(a) &= -\frac{2}{\pi}\int_0^{\infty} \cos\left\{a \coth\varphi\right\} d\varphi . \end{aligned} \right. \tag{9}$$

Then, interpret the above.



Let us return to $\phi = ue^{i\omega t}$ where u is a solution of the Helmholtz equation; hence, ϕ is a (complex) solution of the wave equation, to which a volume energy density* can be attached:

$$\frac{1}{2}\rho\left\{(\nabla\phi)^2 + c^{-2}\left(\frac{\partial\phi}{\partial t}\right)^2\right\}, \tag{10}$$

and a surface density vector of the energy flux

$$-\rho\frac{\partial\phi}{\partial t}\nabla\phi, \tag{11}$$

from which, using the conventional notations, we obtain /II.8

$$\frac{1}{2}\frac{\partial}{\partial t}\iiint_{\mathcal{D}}\left\{(\nabla\phi)^2 + c^{-2}\left(\frac{\partial\phi}{\partial t}\right)^2\right\} dx + \iint_{\Sigma}\frac{\partial\phi}{\partial t}\frac{d\phi}{dn}\, dS = 0, \tag{12}$$

---

* $(\nabla\phi)^2 = \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 + \left(\frac{\partial\phi}{\partial z}\right)^2.$

where the normal derivative is evaluated in a direction pointing toward the interior of $\mathfrak{D}$. If the function ϕ is a complex solution of the wave equation, then ϕ^*, the known complex conjugate, is also a solution of the wave equation and proves eq.(12), just as is done by $\phi + \phi^*$; from this, we derive

$$\frac{\partial}{\partial t}\iiint_{\mathfrak{D}}\left\{\nabla\phi\cdot\nabla\phi^* + c^{-2}\frac{\partial\phi}{\partial t}\frac{\partial\phi^*}{\partial t}\right\}dx + \iint_{\Sigma}\left(\frac{\partial\phi}{\partial t}\frac{d\phi^*}{dn} + \frac{\partial\phi^*}{\partial t}\frac{d\phi}{dn}\right)dS = 0. \tag{13}$$

Let us now place ourselves into a permanent regime, established with $\phi = e^{i\omega t}u$, $\phi^* = e^{-i\omega t}u^*$. Equation (13) will then yield

$$\iint_{\Sigma}\left(u^*\frac{du}{dn} - u\frac{du^*}{dn}\right)dS = 0, \tag{14}$$

which is nothing else but eq.(1), since $u^*\Delta u - u\Delta u^* = 0$; however, from eq.(13) another interesting relation can be derived. In fact, eq.(13) is still valid - for the same reason - if ϕ and ϕ^* are two arbitrary solutions of the wave equation; thus, we can use $\phi_1 = ue^{i\omega t}$, $\phi_2 = u^*e^{i\omega t}$ where u^* is the complex conjugate of u, which yields*

$$\iiint_{\mathfrak{D}}\left\{|\nabla u|^2 - k^2|u|^2\right\}dx + \frac{1}{2}\iint_{\Sigma}\left(u\frac{du^*}{dn} + u^*\frac{du}{dn}\right)dS = 0. \tag{15}$$

It is possible to condense eqs.(14) and (15) into a single entry identity:

/II,9

$$\iiint_{\mathfrak{D}}\left(|\nabla u|^2 - k^2|u|^2\right)dx + \iint_{\Sigma} u\frac{du^*}{dn}\,dS = 0, \tag{16}$$

from which eqs.(14) and (15) are obtained, by taking there the imaginary and real parts.

Problem 4: If u is a solution of the Helmholtz equation, we will have, with u^* being the complex conjugate of u,

$$\nabla\cdot(u\nabla u^*) = |\nabla u|^2 - k^2|u|^2. \tag{17}$$

---

* Here, $|\nabla u|^2 = \left|\frac{\partial u}{\partial x}\right|^2 + \left|\frac{\partial u}{\partial y}\right|^2 + \left|\frac{\partial u}{\partial z}\right|^2$, involving moduli.

Derive eq.(16) from this.

It should be mentioned that eq.(16) is still valid at complex k, provided that k is replaced by k^* and also, without moduli, in the form of

$$\iiint_{\mathcal{D}} \left\{ (\nabla u)^2 - k^2 u^2 \right\} dx + \iint_{\Sigma} u \frac{du}{dn} \, ds = 0 . \tag{18}$$

Let us return to eqs.(13) and (14), assuming Φ to be complex and Φ^* to be its complex conjugate; then, eqs.(13) and (14) can be interpreted on replacing ϕ by $\frac{1}{2}(\phi + \phi^*) = \frac{1}{2}(u\, e^{i\omega t} + u^* e^{-i\omega t})$; under these conditions, the energy volume density becomes

$$\frac{1}{8}\rho \left\{ \left((\nabla u)^2 - k^2 u^2\right) e^{2i\omega t} + \left((\nabla u^*)^2 - k^2 u^{*2}\right) e^{-2i\omega t} \right\} + \boxed{\frac{1}{4}\rho \left\{ |\nabla u|^2 + k^2 |u|^2 \right\}} , \tag{19}$$

whereas the surface density vector of the energy flux will be

$$\frac{1}{4}\rho i k c \left\{ u^* \nabla u^* \, e^{-2i\omega t} - u \nabla u \, e^{2i\omega t} \right\} + \boxed{\frac{1}{4}\rho i k c \left(u^* \nabla u - u \nabla u^* \right)} . \tag{20}$$

Each of these expressions, if it is real, comprises a periodic portion of pulsations 2ω and of zero mean as well as a portion independent of time which remains isolated if the mean is taken over a period which has been encircled by us with a broken line. In speaking of the volume energy density and of the surface density vector of the energy flux in a pulsating sound field, we always mean these boxed terms. Let us note that $(\nabla u)^2$ is the complex square $\nabla u \cdot \nabla u$ of the vector ∇u, whereas $|\nabla u|^2$ is the square of the modulus, i.e., $\nabla u \cdot \nabla u^*$. /II,10

2.3 Results on the Behavior at Infinity

As shown above, if u proves the Helmholtz equation, with possibly a second term vanishing identically in the neighborhood of infinity and if this solution corresponds to the oscillatory state of the steady-state regime of an acoustic phenomenon, started at an epoch quite far in the past, it must be expected that u behaves in $A \dfrac{e^{-ik|x|}}{x}$ for $|x| \to \infty$. However, we have seen that there are also

solutions which behave differently, if only in $A\,\dfrac{e^{+ik|x|}}{|x|}$.

Problem 5: Let ω be a unit vector and demonstrate that the expression

$$u(x) = \frac{ik}{2\pi} \iint I(\omega) \exp\left\{ik\left(\alpha(\omega) + x\cdot\omega\right)\right\} d\omega, \tag{1}$$

where $I(\omega)$ and $\alpha(\omega)$ are regular functions, is itself a regular function in any space which is a solution of the Helmholtz equation.

Problem 6: On a sphere of the radius 1, denoted by Ω, let us differentiate the hemispheres Ω_x^+ and Ω_x^-, characterized by $\omega\ x > 0$ and $\omega \cdot x < 0$; eq.(21) can then be brought to the following form:

$$u(x) = u^+(x) + u^-(x) \tag{2}$$

with /II,11

$$u^{(+,-)}(x) = \frac{ik}{2\pi} \iint_{\Omega_x^{(+,-)}} I(\omega)\ e^{ik(\alpha(\omega) + x\cdot(\omega))}\, d\omega\ . \tag{3}$$

Demonstrate that, if I and α have constant values $I^{(+,-)}$ $\alpha^{(+,-)}$ on the hemispheres $\Omega_x^{(+,-)}$, it follows that

$$u^{(+,-)}(x) = \pm\ I^{(+,-)} e^{ik\alpha^{(+,-)}} \frac{e^{\pm ik|x|} - 1}{|x|}\ . \tag{4}$$

What value might $u(x)$ have if I as well as α is constant over any Ω?

Problem 7: Let $J(x, z)$ be the mean value of $I(\omega)\ e^{ik\alpha(\omega)}$ on $x \cdot \omega = |x|\ z = rz$ and demonstrate that

$$\begin{aligned}(ik)^{-1} u^{(+,-)}(x) = {} & \pm \frac{1}{2} \int_0^1 \left\{J(x, \pm z) - J\left(x, \pm\left[z - \tfrac{\pi}{k|x|}\right]\right)\right\} e^{\pm ik|x|z}\, dz \\ & \mp \frac{1}{2} \int_0^{\pi/k|x|} J\left(x, \pm\left[z - \tfrac{\pi}{kx}\right]\right) e^{\pm ik|x|z}\, dz \\ & \pm \frac{1}{2} \int_1^{\frac{\pi}{k|x|}+1} J\left(x, \pm\left[z - \tfrac{\pi}{kx}\right]\right) e^{\pm ik|x|z}\, dz\ .\end{aligned} \tag{5}$$

Derive from this that, if $|x| \to \infty$, we will asymptotically have

$$u(x) \cong \frac{1}{|x|}\left\{ I\left(\frac{x}{|x|}\right) e^{ik\left(\alpha\left(\frac{x}{|x|}\right)+|x|\right)} - I\left(-\frac{x}{|x|}\right) e^{ik\left(\alpha\left(-\frac{x}{|x|}\right)-|x|\right)} \right\}.$$

Therefore, it is useful to construct theorems on the a priori asymptotic behavior of the solutions of the Helmholtz equation in the neighborhood of infinity. Thus, let $u(x)$ be a twice continuously differentiable solution of the Helmholtz equation, for $|x| > r_0$ in three-dimensional space; it then can be stated that this is a Helmholtz function in the neighborhood of infinity. The mean $\mathfrak{M}_r^x(u)$ proves the equation

$$\frac{\partial^2 r\, \mathfrak{M}_r^x(u)}{\partial r^2} + k^2 r\, \mathfrak{M}_r^x(u) = r\, \mathfrak{M}_r^x\left(\Delta u + k^2 u\right) = 0, \tag{6}$$

provided that the sphere of center x and of radius r contains, in its interior, the sphere with the origin as center and the radius r_0. As a consequence, for any x, it is possible to find r such that, for $r \geq r_1$, we have /II,12

$$\mathfrak{M}_r^x(u) = \frac{1}{2ik}\frac{e^{ikr}}{r} u_I(x) - \frac{1}{2ik}\frac{e^{-ikr}}{r} u_{II}(x), \tag{7}$$

where $u_I(x)$ and $u_{II}(x)$ do not depend on r. In addition, $u_I(x)$ and $u_{II}(x)$ are defined in the entire space and are solutions of the Helmholtz equation; it can be stated that these are integral Helmholtz functions. In fact, for any x, we have

$$\left\{ \begin{aligned} u_I(x) &= e^{-ikr}\left(\frac{\partial}{\partial r} + ik\right)\left(r\, \mathfrak{M}_r^x(u)\right), \\ u_{II}(x) &= e^{ikr}\left(\frac{\partial}{\partial r} - ik\right)\left(r\, \mathfrak{M}_r^x(u)\right), \end{aligned} \right. \tag{8}$$

under the condition that r is taken sufficiently large, and the mentioned result is due to the fact that $\Delta + k^2$ permutes with $e^{\pm ikr}\left(\frac{\partial}{\partial r} \pm ik\right) r\mathfrak{M}_r^x$, since Δ does not extend over the variable r. For the functions u_I and u_{II}, which are integral Helmholtz functions, the resolution of eq.(7) is valid for any r and, specifically, it is then possible to set r = 0. From this, it follows that the following is valid:

$$\begin{cases} \mathfrak{M}_r^*(u_I) = u_I(x)\,\dfrac{\sin(kr)}{kr}, \\ \mathfrak{M}_r^*(u_{II}) = u_{II}(x)\,\dfrac{\sin(kr)}{kr}. \end{cases} \tag{9}$$

If we then derive

$$\begin{cases} V_I(x) = u(x) - u_I(x), \\ V_{II}(x) = u(x) - u_{II}(x), \end{cases} \tag{10}$$

we will obtain

$$\begin{cases} \mathfrak{M}_r^*(V_I) = \dfrac{u_I - u_{II}}{2ki}\,\dfrac{e^{-ikr}}{r}, \\ \mathfrak{M}_r^*(V_{II}) = \dfrac{u_I - u_{II}}{2ki}\,\dfrac{e^{ikr}}{r}. \end{cases} \tag{11}$$

Radiant Helmholtz (wave) function: /II,13

$$u(x) \text{ radiant, if } u_I \equiv 0. \tag{12}$$

Antiradiant Helmholtz (wave) function:

$$u(x) \text{ antiradiant, if } u_{II} \equiv 0. \tag{13}$$

Theorem 2: Any solution of the Helmholtz equation in the neighborhood of infinity is decomposable in one and only one manner, namely, into the sum of an integral Helmholtz function and a radiant Helmholtz function. The decomposition is as follows:

$$\begin{cases} u_I(x) = e^{-ikr}\left(\dfrac{\partial}{\partial r} + ik\right)\left(r\,\mathfrak{M}_r^*(u)\right), & a) \\ V_I(x) = -\dfrac{1}{4\pi}\displaystyle\iint_S \left\{\dfrac{e^{-ikR}}{R}\dfrac{du}{dn} - u\,\dfrac{d}{dn}\,\dfrac{e^{-ikR}}{R}\right\} dS, & b) \\ u(x) = u_I(x) + V_I(x). & c) \end{cases} \tag{14}$$

In eq.(14b), S is any closed surface, completely contained within the neighborhood of infinity where the function u is a solution of the Helmholtz equation and where $\frac{d}{dn}$ represents the normal derivative in a direction pointing toward the neighborhood in question, while R denotes the distance of the point $\boldsymbol{x}$ from the moving point on S.

It has been shown above that this decomposition exists, but now let us prove that it is unique. Let $u_I' + v_I'$ be still another decomposition and let us write

$$u_I - u_I' = v_I' - v_1 = w, \tag{15}$$

where w is an integral Helmholtz function, so that

$$w(x)\frac{\sin(kr)}{kr} = \mathcal{M}_r^*(w) \ . \tag{16}$$

However, $w(\boldsymbol{x})$ is also a radiant function which is not compatible with eq.(16) unless $w \equiv 0$. Let us now apply eq.(2.2) to u by using, for Σ, the combination of S and of a sphere centered in $\boldsymbol{x}$ of a radius r_1, containing S in its interior, thus yielding /II,14

$$\begin{aligned} u(x) = &-\frac{1}{4\pi}\iint_S\left\{\frac{e^{-ikR}}{R}\frac{du}{dn} - u\frac{d}{dn}\frac{e^{-ikR}}{R}\right\}dS + r_1 e^{-ikr_1}\frac{\partial}{\partial n_1}\mathcal{M}_{r_1}^*(u) \\ &+ \mathcal{M}_{r_1}^*(u)\, e^{-ikr_1} + ikr_1\, e^{-ikr_1}\mathcal{M}_{r_1}^*(u) \\ = &-\frac{1}{4\pi}\iint_S\left(\frac{e^{-ikR}}{R}\frac{du}{dn} - u\frac{d}{dn}\frac{e^{-ikR}}{R}\right)dS + u_I(x), \end{aligned} \tag{17}$$

which proves eq.(14b).

The Helmholtz function which was obtained in Section 1.2 and which characacterizes the steady state of a forced oscillation system, obviously is a radiant function since its integral portion is zero; conversely, the solution, restricted to Problem 5, is an integral Helmholtz function since its radiant part is zero.

Problem 8: Demonstrate that any Helmholtz function, in the neighborhood of infinity, admits a convergent series of the following form:

$$u = e^{-ik|x|} \sum_{n=1}^{\infty} \frac{Y_n\left(\frac{x}{|x|}\right)}{|x|^n}, \tag{18}$$

where $Y_n(\omega)$ is a combination of spherical harmonics of an order less than or equal to n. Here, we call "spherical harmonic of order n" a function $\mathfrak{H}_n(\omega)$, defined on the sphere of radius 1, such that $\frac{1}{|x|^n} \mathfrak{H}_n\left(\frac{x}{|x|}\right)$ becomes a solution of the Laplace equation.

2.4 Radiation Conditions

/II,15

Each time that a problem of searching for a Helmholtz function is posed in a domain that contains the point at infinity, it is necessary - to obtain uniqueness of the possible solution - to impose a condition at infinity. In numerous cases, it is prescribed that the function be radiant. Using suitably selected limit conditions at a finite distance, it could be shown that the solution, if it exists at all, is completely determined. It should be noted in this respect that, if the function is a solution of the Helmholtz equation in the entire space, the only limit condition possible is at infinity; for this reason, if it is stipulated that the solution be radiant, this solution will be well defined since it then will be identically zero.

Consequently, we will look for various ways to express that a Helmholtz function is radiant.

Theorem 3: Any radiant Helmholtz function proves the Sommerfeld radiation condition: If the point x is located on a radius emerging from a fixed center, in such a manner that the distance r to the center tends toward infinite, we will have the following expression, uniformly distributed over all directions:

$$\lim_{r \to \infty} \left\{ r\left(\frac{\partial u}{\partial r} + iku\right) \right\} = 0 . \tag{1}$$

Conversely, if a Helmholtz function proves the Sommerfeld radiation condition, it will be radiant. The Sommerfeld condition, in addition, can be substituted by the following condition

$$\lim_{r \to \infty} \left\{ r^2 \,\mathfrak{M}^{*}_{r}\left(\left|\frac{\partial u}{\partial r} + iku\right|^2\right) \right\} = 0 \tag{2}$$

for a fixed center. We then pose $\frac{\partial u}{\partial r} = \frac{\partial}{\partial r} u(x_0 + r\,\omega)$.

/II,16

Let us make use of the result of Problem 8:

$$u = e^{-ikr} \sum_{n=1}^{\infty} \frac{Y_n}{r^n} , \tag{3}$$

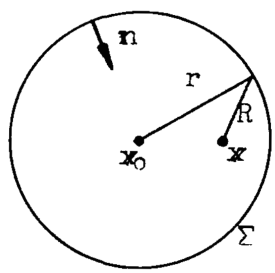
It is obvious that it is possible to differentiate term by term, so that

$$r\left(\frac{\partial u}{\partial r} + iku\right) = - e^{-ikr} \sum_{n=1}^{\infty} \frac{n Y_n}{r^n} , \tag{4}$$

Here, the second term tends toward zero at $r \to \infty$; similarly,

$$r^2 \mathfrak{M}^0_r \left(\left| \frac{\partial u}{\partial r} + iku \right|^2 \right) = \frac{Y_1^2}{r^2} + O\left(\frac{1}{r^3}\right) . \tag{5}$$

Let us demonstrate now that eq.(2) implies that u is radiant. Let us consider the sphere Σ of a center $\boldsymbol{x}_0$ and a radius r and let us attempt to evaluate $u_I(\boldsymbol{x})$ at a point $\boldsymbol{x}$ interior to this sphere, by assuming that $\boldsymbol{x}$ is located in the



region in which u is a Helmholtz function. For this, let us introduce the surface S which enters in eq.(3.14b), by assuming that S is located at the interior of Σ and let us then apply eq.(2.2) to the domain comprised between S and Σ, thus yielding

$$u_I(\boldsymbol{x}) = \frac{1}{4\pi} \iint_{\Sigma} \left\{ \frac{\partial u}{\partial r} \frac{e^{-ikR}}{R} - u \frac{\partial}{\partial r} \frac{e^{-ikR}}{R} \right\} dS , \tag{6}$$

since the contribution of S is exactly equal to the radiant portion v_I of u. Here, $R = |\boldsymbol{\xi} - \boldsymbol{x}|$ where $\boldsymbol{\xi}$ is a moving point of Σ. Equation (6) is written as

$$u_I(x) = \frac{1}{4\pi}\iint_\Sigma \left(\frac{\partial u}{\partial r} + iku\right)\frac{e^{-ikR}}{R}\,ds + \frac{1}{4\pi}\iint_\Sigma u\left\{\frac{\partial R}{R\partial r} + ik\left(\frac{\partial R}{\partial r} - 1\right)\right\}\frac{e^{-ikR}}{R}\,ds = I_1 + I_2 \quad (7)$$

and we will demonstrate that I_1 and I_2 tend separately to zero as soon as $r \to \infty$, which would mean that $u_I \equiv 0$ since u_I does not depend on r. It is obvious /II,17/ that $\frac{r}{R} e^{-ikR}$ remains bounded in the vicinity of $r = \infty$ - it is assumed that x_0 and x are fixed - so that, according to Schwartz's inequality, we will have the following expression in the vicinity of $r = \infty$:

$$I_1 \le 2\left\{r^2\, \mathfrak{M}_r^{x_0}\left(\left|\frac{\partial u}{\partial r} + iku\right|^2\right)\right\}^{\frac{1}{2}}, \quad (8)$$

and I_1 definitely tends to zero as a consequence of eq.(2). To eliminate any uncertainty, let us specify that the Schwartz inequality is applied to $\frac{1}{4\pi r}\iint_\Sigma \left(\frac{\partial u}{\partial r} + iku\right)\frac{r}{R}e^{-ikR}\,dS$. Now, if I_2 is written in the form of $\frac{1}{4\pi}\iint_\Sigma \frac{u}{r^2}\left(\frac{r}{R}\right)^2\left\{\frac{\partial R}{R\partial n} + ik\left(\frac{\partial R}{\partial r} - 1\right)\right\} R\, e^{-ikR}dS$ and if it is noted that $\left(\frac{r}{R}\right)^2 \frac{\partial R}{\partial r} \cdot e^{-ikR}$ is augmented in modulus by a constant in an infinity neighborhood, then a re-application of the Schwartz inequality will demonstrate that I_2 is augmented by const $\{\mathfrak{M}_r^{x_0}(|u|^2)\}^{1/2}$ and thus also tends to zero. It remains to be mentioned that, if eq.(1) takes place uniformly in all directions, [here, $f(r\,\omega) \to 0$ (uniformly) if we can find $g(r) \to 0$ such that $|f(r\,\omega)| \le g(r)$ for any $|\omega| = 1$], then eq.(2) will take place.

2.5 Potentials of the Single and Double Layer. Volume Potential

2.5.1 Potential of Volume

The expression

$$\mathcal{V}(x; f, \mathfrak{D}) = \frac{1}{4\pi}\iiint_{\mathfrak{D}} f(\xi)\,\frac{e^{-ik|x-\xi|}}{|x-\xi|}\,d\xi \quad (1)$$

is known as the Helmholtz volume potential. The domain $\mathfrak{D}$ is assumed as bounded

and the function f is continuous.

Theorem 4: If the function $f(\mathbf{x})$ is a continuous Hölderian function, i.e., if we have $|f(\mathbf{x}_1) - f(\mathbf{x}_2)| < \text{const}\ |\mathbf{x}_1 - \mathbf{x}_2|^\alpha$ $0 < \alpha \le 1$, then $\mathfrak{D}(\mathbf{x}; f, \mathfrak{D})$ is twice continuously differentiable, yielding /II,18

$$\begin{cases} \Delta \mathcal{V} + k^2 \mathcal{V} = -f, & \text{within } \mathfrak{D} \\ \Delta \mathcal{V} + k^2 \mathcal{V} = 0, & \text{outside of } \mathfrak{D} \end{cases} \tag{2}$$

where $\mathfrak{B}$ is a radiant Helmholtz function outside of $\mathfrak{D}$. On replacing $e^{-ik|\mathbf{x} - \xi|}$ by $e^{ik|\mathbf{x} - \xi|}$, we obtain an antiradiant function.

It is useful to note that the differentiability of $\mathfrak{B}$, in a point $\mathbf{x}$, will involve the properties of f only in the vicinity of this same point $\mathbf{x}$.

Demonstration of this theorem requires the use of the Newtonian potential theory, which we will not give here. However, we should mention that, if f is indefinitely differentiable - it is sufficient that it is continuously differentiable - and if it has a compact base (f is zero outside of a sphere), then $\mathfrak{D}$ can be replaced by the entire space and eq.(1) can be considered as being a convolution

$$\mathcal{V} = f * \frac{e^{-ik|\mathbf{x}|}}{|\mathbf{x}|}, \tag{3}$$

such that it can be readily differentiated in the sense of distributions

$$\Delta \mathcal{V} + k^2 \mathcal{V} = f * \left\{ \Delta \frac{e^{-ik|\mathbf{x}|}}{|\mathbf{x}|} + k^2 \frac{e^{-ik|\mathbf{x}|}}{|\mathbf{x}|} \right\} = -f * \delta = -f \tag{4}$$

by making use of Problem 1. More intuitively, $\dfrac{e^{-ikr}}{4\pi r}$ is a solution of the Helmholtz equation which, in $r = 0$, presents the same singularity as the fundamental solution $\dfrac{1}{4\pi r}$ of the Laplace equation. /II,19

Problem 9: Suppose that f is continuous and has a compact base; apply eq.(2.1), by using for $\mathbb{D}$ the exterior $\mathfrak{D}_\varepsilon$ of a sphere with center $\mathbf{x}$ and radius ϵ, while taking for V the function of ξ, $\dfrac{1}{4\pi}\ \dfrac{e^{-ik|\mathbf{x} - \xi|}}{|\mathbf{x} - \xi|}$ where u is the solution, assumed to exist, of $\Delta u + k^2 u = -f$ which is radiant at infinity, and then cause ϵ to tend toward zero. Demonstrate that this will yield the following theorem of representation: If u is twice continuously differentiable in the

entire space and if it is a radiant Helmholtz function in the neighborhood of infinity, one would obtain

$$u(x) = -\frac{1}{4\pi}\iiint \left\{\Delta u(\xi) + K^2 u(\xi)\right\} \frac{e^{-iK|x-\xi|}}{|x-\xi|}\, d\xi. \tag{5}$$

Explain why this does not constitute a proof of theorem 4.

2.5.2 Potential of the Single Layer

We will state that a surface Σ is regular if the normal unit vector $\mathbf{n}(\mathbf{x})$ depends continuously and differentiably on the point $\mathbf{x}$ on Σ. In practical use, it is necessary to consider regular surfaces by sections, representing edges or conical points.

Consider a function $f(\mathbf{x})$, continuously dependent on the point $\mathbf{x}$ on Σ, whic vanishes sufficiently rapidly at infinity if Σ tends toward infinite. For certain cases, it must be assumed that $f(\mathbf{x})$ may become infinite along certain lines or at certain points - which generally represent the edges or the conic points of Σ - but remains absolutely integrable on Σ.

The following expression: /II,

$$\mathfrak{J}^{I}(x; f, \Sigma) = \frac{1}{2\pi}\iint_{\Sigma} f(\xi)\, \frac{e^{-iK|x-\xi|}}{|x-\xi|}\, dS_{\xi}, \tag{6}$$

is known as the single-layer potential.

Theorem 5: The single-layer potential $\mathfrak{J}^{I}(\mathbf{x};\, f,\, \Sigma)$ is a solution of the Helmholtz equation at any point in space which is not located on Σ; in fact, this potential even is an indefinitely differentiable solution of this equation (being analytic!). If Σ is a complete integer at a finite distance, then $\mathfrak{J}^{I}(\mathbf{x};\, f,\, \Sigma)$ is a radiant Helmholtz function in the neighborhood of infinity.

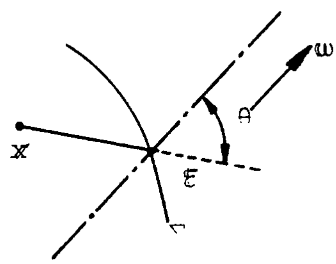
The proof of this theorem is trivial, by noting that, if $\mathbf{x}$ is not located on Σ, differentiation can be made under the $\iint$ sign with respect to $\mathbf{x}$ as many times as desired. If Σ does not tend to infinity, then $\mathfrak{J}^{I}$ can be expanded in a convergent series in powers of $|\mathbf{x}|^{-1}$, with the first term being

$$\left(\frac{1}{2\pi}\iint_{\Sigma} f(x)\, dS_{\xi}\right) \frac{e^{-iK|x|}}{|x|}, \tag{7}$$

which makes it obvious that a radiant function is involved here.

Theorem 6: The single-layer potential is continuous on crossing of Σ at any point at a finite distance which, at the same time, is a regularity point of Σ in whose vicinity f remains bounded. Let us assume that f is a continuous Hölderian in the vicinity of a point x_0 of Σ, with Σ being regular in this vicinity; then, the first partial derivatives of $\mathfrak{J}^I$ tend toward limiting values whenever x tends toward a point of the neighborhood in question while still remaining on the same side of Σ, by taking a path which admits, at the remote point, a tangent not contained in the plane tangent to Σ. /II,21

Let us isolate, around x_0 on Σ, a small disk of radius ϵ - a quasi-plane disk - and let us write $\Sigma = \Sigma_\varepsilon + \Delta_\varepsilon$ where Δ_ε is the disk in question. The integral over Σ_ε is continuous in x_0 while the integral over Δ_ε is augmented by const ϵ so that, since ϵ is arbitrary, the expression $|\mathfrak{J}^I(x) - \mathfrak{J}^I(x_0)|$ can be



made arbitrarily small by first making (in this order!) ϵ small and then also making $|x - x_0|$ small. Thus, the single-layer integral is continuous at the crossing of Σ. Let us now search for the derivative $\omega \cdot \nabla \mathfrak{J}^I(x)$ of the single-layer potential in a direction ω for a point x located on Σ. Let, in accordance with the accompanying diagram, $\theta(x, \xi)$ be the angle made by the vector ω with the direction $\xi - x$, yielding

$$\begin{cases} \omega \cdot \nabla \mathfrak{J}^I = I(x;\omega) + J, \\ I(x,\omega) = \dfrac{1}{2\pi} \displaystyle\iint_\Sigma \dfrac{f(\xi)\cos[\theta(x,\xi)]}{|x-\xi|^2}\, dS_\xi, \end{cases} \tag{8}$$

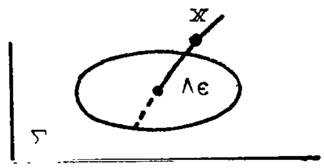
where J is an integral of the same type as the single-layer potential (i.e., with $|x - \xi|$ in the denominator and with a bounded numerator), which is thus continuous on crossing of Σ. Now, it is obvious that

$$I(x) = \frac{f(x)}{2\pi} \iint_\Sigma \frac{\cos[\theta(x,\xi)]}{|x-\xi|^2}\, dS_\xi + \frac{1}{2\pi} \iint_\Sigma \frac{[f(x)-f(\xi)]\cos\theta(x,\xi)}{|x-\xi|^2}\, dS_\xi, \tag{9}$$

since, if x is sufficiently close to Σ, it is always possible to extend the definition of f in such a manner that it becomes a Hölderian continuous function of the point in space about Σ - for example, f = const on a normal to Σ. The second integral in eq.(9) is treated like the single-layer potential, except that this time the contribution of Λ_ε is augmented by const ε^{a}, with the /II,22
point x being located on the normal to the center of the disk or, expressed more generally, the center of the disk located at the foot of the path taken by x in tending toward Σ (a minor technical detail is involved here, taking into consideration that x emerges from the plane of the disk!). Finally, because of

$$\frac{1}{2\pi}\iint_{\Sigma}\frac{\cos[\theta(x,\xi)]}{|x-\xi|^2}\,dS_\xi = \frac{1}{2\pi}\,\omega\cdot\nabla\iint_{\Sigma}\frac{dS_\xi}{|x-\xi|}, \tag{10}$$

the proof of theorem 6 is reduced to the analogous theorem for the single-layer Newtonian potential. Let us recall the proof of this latter theorem. We write $\Sigma = \Sigma_\varepsilon + \Lambda_\varepsilon$ where Σ_ε represents no problem. For Λ_ε, a technical part is required which consists in demonstrating that, at an error tending to zero with ε,



it is possible to replace the integral over Λ_ε by the integral over the projection of Λ_ε onto the plane tangent to Σ at x_0, which is the foot of the path taken by x in tending toward Σ. In omitting this part, we return to the study of

$$I(x_1,x_2,x_3) = \frac{1}{2\pi}\iint_{\xi_1^2+\xi_2^2<\varepsilon^2}\frac{\cos[\theta(x_1,x_2,x_3;\xi_1,\xi_2)]}{(x_1-\xi_1)^2+(x_2-\xi_2)^2+x_3^2}\,d\xi_1\,d\xi_2 \tag{11}$$

with

$$\cos\theta = \frac{\omega_1(\xi_1-x_1)+\omega_2(\xi_2-x_2)-\omega_3 x_3}{\{(x_1-\xi_1)^2+(x_2-\xi_2)^2+x_3^2\}^{3/2}}, \tag{12}$$

such that we obtain

$$I = \omega_1 I_1 + \omega_2 I_2 - \omega_3 I_3 \tag{13}$$

and, obviously,

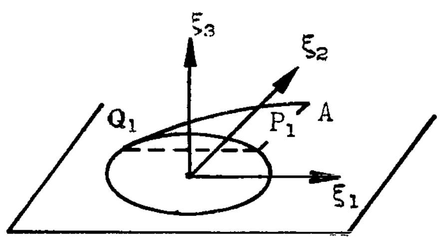
$$I_{1,2} = -\frac{1}{2\pi}\int_{n}\left(\frac{1}{AP_{1,2}} - \frac{1}{AQ_{1,2}}\right) d\xi_{2,1} \tag{14}$$

where P_1 and Q_1 are points of the rim Γ_ε of the disk Δ_ε, located on $\xi_2 =$ /II,23 $=$ const where A is the tip of the coordinates x_1, x_2, x_3. It is thus obvious that I_1 and I_2 tend to zero as soon as x_1, x_2, x_3 simultaneously tend to zero. This leaves I_3 to be treated:

$$I_3 = \frac{1}{2\pi}\iint_{\sqrt{\xi_1^2+\xi_2^2}<\varepsilon} \frac{x_3}{\{(x_1-\xi_1)^2 + (x_2-\xi_2)^2 + x_3^2\}^{3/2}}\, d\xi_1\, d\xi_2 \tag{15}$$

$$= J_3 - K_3,$$

where J_3 is the integral analogous to I_3 extending over the entire plane, if K_3 is extended to the exterior of the disk $\xi_1^2 + \xi_2^2 < \epsilon^2$. It is obvious that $K_3 \to 0$



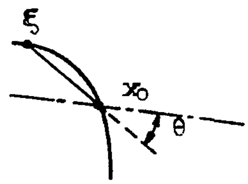
as soon as x_1, x_2, and x_3 simultaneously tend to zero, but that

$$J_3 = \int_0^\infty \frac{x_3\, \rho\, d\rho}{(\rho^2 + x_3^2)^{3/2}} = \frac{x_3}{|x_3|}. \tag{16}$$

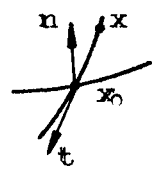
Let us now return to Σ and let us consider the integral

$$\frac{1}{2\pi}\iint_{\Sigma_\varepsilon} \frac{\rho(\xi)\cos[\theta(x_0 - \xi)]}{|x_0 - \xi|}\, dS_\xi = K(\varepsilon) = \omega_1 K_1(\varepsilon) + \omega_2 K_2(\varepsilon) - \omega_3 K_3(\varepsilon) \tag{17}$$

where the point x_0 is located on Σ and where Σ_ε, as above, is the part of Σ exterior to the disk Δ_ε with ω_1, ω_2, ω_3 being the three components of ω at an orthogonal reference point selected in such a manner that $\omega_3 = 0$ if ω is located within the plane tangent to x_0. It will be demonstrated separately that $K_1(\epsilon)$, $K_2(\epsilon)$, and $K_3(\epsilon)$ tend toward finite limits as soon as $\epsilon \to 0$. For K_3, this



results from the fact that a convergent integral is involved here whereas, for K_1 and K_2, it results from the fact that the contributions of the order ϵ^{-1} are simultaneously positive and negative and behave exactly in accordance with /II,24 the mechanism indicated in eq.(14).



Theorem 7: Let $\mathbf{n}$ be the normal unit vector in $\mathbf{x}_0$ at Σ and let us assume that $\mathbf{x}$ tends toward $\mathbf{x}_0$ by taking a path that admits, in $\mathbf{x}_0$, a tangent vector $\mathbf{t}$; let us pose

$$\varepsilon = -\operatorname{sgn} \mathbf{t} \cdot \mathbf{n}, \tag{18}$$

so that $\epsilon = +1$ where $\mathbf{x}$ tends to $\mathbf{x}_0$ while remaining on the same side as $\mathbf{n}$ with respect to Σ, whereas $\epsilon = -1$ when $\mathbf{x}$ remains on the other side. Under these conditions, if $\mathbf{x}_0$ is a regularity point of Σ and if $f(\mathbf{x})$ is a continuous Hölderian on Σ in the vicinity of $\mathbf{x}_0$, then the gradient $\nabla \mathfrak{J}^{I}(\mathbf{x}; f, \Sigma)$, as soon as $\mathbf{x}$ tends to $\mathbf{x}_0$, will tend toward a limiting value equal to

$$-\varepsilon\, \mathbf{n}\, f(\mathbf{x}_0) + \frac{1}{2\pi} \iint_{\Sigma} f(\boldsymbol{\xi})\, \nabla_{x} \frac{e^{-iK|\mathbf{x}_0 - \boldsymbol{\xi}|}}{|\mathbf{x}_0 - \boldsymbol{\xi}|}\, dS_{\xi}, \tag{19}$$

where the improper integral is defined as the limit (when $\epsilon \to 0$) of the corresponding integral extending to Σ computed from a disk centered in $\mathbf{x}_0$ and having a radius ϵ.

2.5.3 Potential of the Double Layer

The expression

$$\mathcal{S}^{II}(x; f, \Sigma) = \frac{1}{8\pi} \iint_{\Sigma} f(\xi) \frac{d}{dn_\xi}\left(\frac{e^{-ik|x-\xi|}}{|x-\xi|}\right) dS_\xi , \qquad (20)$$

is known as the double-layer potential. Here, $\frac{d}{dn_\xi} = \mathbf{n} \cdot \nabla_\xi$, using the notations of theorem 7.

Theorem 8: The double-layer potential is an indefinitely differentiable solution - and even an analytic solution! - of the Helmholtz equation, everywhere outside of Σ. If Σ remains a complete integer at a finite distance, this will be a radiant Helmholtz function at infinity. If the surface Σ is regular in x_0 and if f is a continuous Hölderian there, then the double-layer potential will tend toward a limit as soon as x tends to x_0 by taking a path which is not tangent in x_0 to Σ; then, the limit in question will be /II,25

$$\varepsilon f(x_0) + \frac{1}{2\pi} \iint_{\Sigma} f(\xi) \frac{d}{dn_\xi} \frac{e^{-ik|x_0-\xi|}}{|x_0-\xi|} dS_\xi , \qquad (21)$$

where $\epsilon = \pm 1$ has the same meaning as in theorem 7.

2.6 Existence and Uniqueness of the Problem of the Diffraction by a Regular Obstacle

2.6.1 Physical Origin of the Problem

One or several regular obstacles (of regular surface) are assumed to be placed in the presence of sound sources. A sound source is to mean here any physical system capable of producing air vibrations. When we return later to the mathematical treatment, it will obviously be necessary to characterize these sound sources by a schematization. The problem of diffraction consists in determining the mode of action of the obstacles in modifying the propagation of sound.

Let us assume that the sound field develops in accordance with the acoustic equations in a homogeneous medium and is derived from a potential ϕ. On the obstacle - or on one of the obstacles - the velocity of air is $\frac{d\phi}{dn}$ and, if the obstacle has an undeformable and impermeable wall, the following must be written

$$\frac{d\phi}{dn} = w_n \qquad (1)$$

where w_n is the rate of normal displacement of the wall. If the obstacle is permeable, the sound field will subject this obstacle to an (algebraic) /II,26 overpressure $-\rho \frac{\partial \phi}{\partial t}$ which entrains a mass flux $-\rho k \frac{\partial \phi}{\partial t}$ toward the interior; then, the condition to be applied reads

$$\frac{d\phi}{dn} - w_n = K \frac{\partial \phi}{\partial t} \tag{2}$$

with $K > 0$. We will retain

$$\alpha \frac{\partial \phi}{\partial t} - \beta\left(\frac{d\phi}{dn} - w_n\right) = 0 \quad , \tag{3}$$

as the condition on the wall of an obstacle, with

$$\alpha \geqslant 0 \quad , \quad \beta > 0 \; . \tag{4}$$

In the space outside the obstacle, ϕ proves the wave equation

$$\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \Delta \phi = 0 \tag{5}$$

at the initial instant, while ϕ and $\frac{\partial \phi}{\partial t}$ are known since both velocity and pressure are known. It then remains to define the schematization to be used in the region occupied by the sound sources. A full discussion is almost impossible here since the generation of sound is a phenomenon of extreme complexity. An occasionally used method is to write $\frac{1}{c^2} \frac{\partial^2 \phi}{\partial t^2} - \Delta\phi = F(t, \mathbf{x})$ where F differs from zero in the region occupied by the sources, and to consider F as known. It is also possible to envelop the sources by a surface S, considering only the field outside of S and assuming that the action of the sources is schematized by the value of the velocity and of the pressure on S, as a function of position and time. This reduces to using ϕ and $\frac{d\phi}{dn}$ on S. Naturally, we do not have the choice of an arbitrary selection of ϕ and $\frac{d\phi}{dn}$, since we are using an artifice here. /II,27

Setting aside the obstacles that diffract the sound and supposing the initial data to be zero, we can represent ϕ in terms of the values taken by ϕ and $\frac{d\phi}{dn}$ on S, in virtue of Kirchhoff's formula

$$\phi = \frac{1}{4\pi} \iint_S \left\{ [\phi] \frac{d}{dn}\left(\frac{1}{r}\right) - \frac{1}{r}\left[\frac{d\phi}{dn}\right] - \frac{1}{cr}\left[\frac{\partial \phi}{\partial t}\right] \right\} dS, \tag{6}$$

where r naturally represents the distance from the point where the value of ϕ is calculated to the moving point on S, while [ ] is the conventional notation for a value taken in t - r/c and $\frac{d}{dn}$ denotes the normal derivative in a direction pointing away from the sources. We are here interested in the case in which the action of the sources produces a steady-state oscillatory motion on S. This presupposes that, on S, we have

$$\lim_{t\to\infty} e^{-i\omega t} \phi = F, \qquad \lim_{t\to\infty} e^{-i\omega t} \frac{d\phi}{dn} = G, \tag{7}$$

in which case eq.(6) shows that an oscillating state is established outside of S, i.e.,

$$\lim_{t\to\infty} e^{-i\omega t} \phi = u, \tag{8}$$

with

$$u(x) = \frac{1}{4\pi} \iint_S \left\{ F(\xi) \frac{d}{dn_\xi} \frac{e^{-ik|x-\xi|}}{|x-\xi|} - G(\xi) \frac{e^{-ik|x-\xi|}}{|x-\xi|} \right\} dS_\xi . \tag{9}$$

It must be considered that, so far as the problem of an effective determination of the sound field produced by a system of sources with given physical characteristics is concerned, eq.(9) yields no solution since F and G are unknown. This is the general type of a representation formula by means of which interesting properties of a general order can be determined, without actually /II,28 having resolved the problem itself. Let us note, in passing, that we have demonstrated the following result: If the sound field, established by a system of sound sources in the absence of any obstacle, develops toward a steady-state oscillating regime, with the pulsation ω, then this regime is characterized by a radiant Helmholtz function. In formulating this result, it is obvious that a representation formula is of considerable interest.

Let us next study the problem of producing sound in the presence of obstacles. It is again possible to use Kirchhoff's formula but it then becomes necessary to add, on the surface S which surrounds the sources, one (or several) surfaces Σ which surround the obstacle (or obstacles) in such a manner that

$$\phi = \phi_S + \phi_\Sigma , \tag{10}$$

$$\phi_{S,\Sigma} = \frac{1}{4\pi}\iint_{S,\Sigma}\left\{[\phi]\frac{d}{dn}\frac{1}{r} - \frac{1}{r}\left[\frac{d\phi}{dn}\right] - \frac{1}{cr}\left[\frac{\partial\phi}{\partial t}\right]\right\} dS \,. \tag{10}$$

Despite the fact that this formula offers no solution, it does permit defining the problem and constructing a certain scheme. Let a be the maximum diameter of S and let δ be that of each of the component parts of Σ, while d is to be the minimum distance from S to Σ. Let us assume that

$$d \gg a \,, \quad d \gg \delta \,, \tag{11}$$

and let us station ourselves in the vicinity of S; this will yield

$$\begin{cases} |\phi|_S = O\left\{\underset{S}{\mathrm{Max}}\left(|\phi| + a\left|\frac{d\phi}{dn}\right| + \frac{a}{c}\left|\frac{\partial\phi}{\partial t}\right|\right)\right\} = O\left(\underset{S}{\mathrm{Max}}\,|\phi|\right) , \\ |\phi_\Sigma| = O\left\{\underset{\Sigma}{\mathrm{Max}}\left(\frac{\delta}{d}|\phi| + \frac{\delta^2}{d}\left|\frac{d\phi}{dn}\right| + \frac{\delta^2}{cd}\left|\frac{\partial\phi}{\partial t}\right|\right)\right\} = \frac{\delta}{d}\, O\left(\underset{\Sigma}{\mathrm{Max}}\,|\phi|\right) , \end{cases} \tag{12}$$

if we assume that ϕ, $\frac{\partial\phi}{\partial t}$, $\frac{d\phi}{dn}$ are regular on S and Σ. This will result /II,29
in

$$\left(\frac{|\phi_\Sigma|}{|\phi_S|}\right)_S = O\left(\frac{\delta}{d}\,\frac{\underset{\Sigma}{\mathrm{Max}}\,|\phi|}{\underset{S}{\mathrm{Max}}\,|\phi|}\right) . \tag{13}$$

If we suppose, which intuitively presents no problem, that $\underset{\Sigma}{\max}|\phi|$ is not greater in order of magnitude than $\underset{S}{\max}|\phi|$, then it is obvious that

$$|\phi_\Sigma| \ll |\phi_S| \,, \qquad \text{in the vicinity of S} \tag{14}$$

which can be expressed in words by stating that the reaction of a sound field, diffracted by the objects, is very weak at the sources. In the schematization, which permits a mathematical formulation of the problem of sound diffraction, this hypothesis of negative feedback always appears in terms of a subjacent hypothesis. In reality, this does not mean that this hypothesis might not occasionally be of doubtful validity but then the problem which we attempt to schematize no longer is that of the diffraction of a pre-established sound field; rather, it is a problem of the production of sound which is frequently much more difficult.

Below, let us use the hypothesis of negative feedback and let ϕ_0 be the sound field created by the sources in the absence of obstacles; in agreement with the above statements, we then can represent the sound field, in the presence of obstacles, by

$$\phi = \frac{1}{4\pi}\iint_S \left\{[\phi_0]\frac{d}{dn}\left(\frac{1}{r}\right) - \frac{1}{r}\left[\frac{d\phi_0}{dn}\right] - \frac{1}{cr}\left[\frac{\partial\phi_0}{\partial t}\right]\right\} ds$$
$$+ \frac{1}{4\pi}\iint_\Sigma \left\{[\phi]\frac{d}{dn}\left(\frac{1}{r}\right) - \frac{1}{r}\left[\frac{d\phi}{dn}\right] - \frac{c}{r}\left[\frac{\partial\phi}{\partial t}\right]\right\} dS, \tag{15}$$

taking into consideration that here we have only an approximate representation, where one of the nonconnectivities consists in that the values of $[\phi]$, /II,30 $\left[\frac{d\phi}{dn}\right]$, $\left[\frac{\partial\phi}{\partial t}\right]$, calculated in accordance with eq.(15) on S, differ from $[\phi_0]$, $\left[\frac{d\phi_0}{dn}\right]$, $\left[\frac{\partial\varphi_0}{\partial t}\right]$ (but very little!). Each of the integrals of eq.(15), however, is a solution of the wave equation, which constitutes the importance of this formula. In addition, the first integral is obviously equal to φ_0, so that we can write

$$\phi = \phi_0 + \psi, \tag{16}$$

where ψ is a solution of the wave equation. From this, the mathematical formulation of the problem for the diffraction of sound is derived: Find a solution ψ of the wave equation, proving zero initial conditions such that $\phi_0 + \psi$ will verify, along the walls of the obstacles,

$$\alpha\left(\frac{\partial\phi_0}{\partial t} + \frac{\partial\psi}{\partial t}\right) - \beta\left(\frac{d\phi_0}{dn} + \frac{d\psi}{dn} - w_n\right) = 0 \tag{17}$$

with nonnegative α and β and with given ϕ_0 and w_n.

In many cases,

$$\lim_{t\to\infty} e^{-i\omega t}\,\phi_0(t,x) = u_0(x) \tag{18}$$

exists; for example, if α and β are constants and if $\lim\limits_{t\to\infty} w_n\, e^{-i\omega t} = \tilde{w}_n$ exists, it can be suspected that

$$\lim_{t\to\infty} e^{-i\omega t}(\phi_0 + \psi) = u_0(x) + u_d(x) \tag{19}$$

will also exist. The problem of determining $u_d(\mathbf{x})$ (of the diffracted sound field) is the problem of the diffraction of Helmholtz waves. It is obvious, according to eq.(15), that the diffracted sound field $u_d(\mathbf{x})$ is radiant at infinity. /II,31

2.6.2 Diffraction of Helmholtz Waves by a Regular Obstacle

Formulation of the problem: Find a solution $u(\mathbf{x})$ of the Helmholtz equation, radiant at infinity and continuously differentiable up to the surface Σ which is assumed to be regular, but twice continuously differentiable outside of Σ and satisfying the condition

$$i\omega\alpha u - \beta\left(\frac{du}{dn} - w\right) = 0, \qquad \omega = kc \tag{20}$$

on Σ, where α and β are nonnegative constants while w is a given function.

Let us assume that two different solutions u_1 and u_2 exist, so that $u = u_1 - u_2$ will prove

$$i\omega\alpha u - \beta\frac{du}{dn} = 0, \tag{21}$$

on Σ. Let us apply the energy identity by using as surface $\Sigma + \Sigma r$ where Σr is a sphere with the radius r, centered on $\mathbf{x}_0$ and surrounding Σ; we then have

$$\mathrm{Im}\iint_{\Sigma_r} u\frac{du^*}{dn}\,dS = -\frac{\alpha}{\beta}\,\omega\iint_{\Sigma} |u|^2\,dS \le 0, \tag{22}$$

where the normal derivative $\frac{d}{dn}$ on Σr is evaluated in a direction pointing toward infinite. It is known that, on Σr,

$$u = \frac{e^{-ikr}}{r}\,V(r,\omega), \qquad \mathbf{x} = \mathbf{x}_0 + r\omega \tag{23}$$

with

$$V = \sum_{n=0}^{\infty} r^{-n} V_n(\omega), \tag{24}$$

we have

$$r^2 u\frac{du^*}{dn} = V\frac{dV^*}{dr} - \frac{|V|^2}{r} + ik|V|^2, \tag{25}$$

so that the inequality (22) will be written as /II,32

$$K \iint_{\Sigma_r} |V|^2 \, ds + \mathrm{Im} \iint_{\Sigma_r} V \frac{dV^*}{dr} \, ds \leq 0. \tag{26}$$

This implies that

$$\iint_{|\omega|=1} |V_0(\omega)|^2 \, d\omega = 0, \tag{27}$$

i.e.,

$$\left\{ \begin{aligned} u &= \frac{e^{-iKr}}{r^2} \, W(r,\omega), \\ W(r,\omega) &= \sum_{n=0}^{\infty} \frac{w_n(\omega)}{r^n}, \end{aligned} \right. \tag{28}$$

from which it follows that

$$\lim_{r\to\infty} \left\{ K^2 r^2 \, \mathfrak{M}_r^{x_0}(|u|^2) + r^2 \, \mathfrak{M}_r^{x_0}(|\nabla u|^2) \right\} = 0, \tag{29}$$

so that the above theorem 10 demonstrates that u ≡ 0 in the neighborhood of infinity whereas theorem 11 demonstrates that u ≡ 0 everywhere outside of Σ.

Theorem 9: The problem of diffraction of Helmholtz waves by a regular obstacle admits at most one solution.

Theorem 10: Let u be a function which is twice continuously differentiable and which proves the Helmholtz equation in the neighborhood of infinity, yielding

$$\liminf_{r\to\infty} \left\{ K^2 r^2 \, \mathfrak{M}_r^{x_0}(|u|^2) + r^2 \, \mathfrak{M}_r^{x_0}(|\nabla u|^2) \right\} > 0, \tag{30}$$

provided that u does not vanish identically in the neighborhood of infinity.

Theorem 11: Let $\mathfrak{L}$ be an elliptic operator with constant coefficients. Let us assume that u vanishes identically in a convex domain Ω_1 if u vanishes identically in the entire convex domain Ω_2, containing Ω_1 where $\mathfrak{L}u$ vanishes. /II,33

The reader will find a demonstration of theorem 11 in the book by Hormander "Linear Partial Differential Operators", Springer 1963, Sect.5.3, Theorem 5.3.3.

Let us prove our theorem 10. With respect to a pole, selected once and for all, let us note $\frac{\partial}{\partial r}$ and ∇_T of the two radial and tangential components of the gradient and, using $w = ru$, let us formulate the expression

$$F(r) = \mathcal{M}_r\left(\left|\frac{\partial w}{\partial r}\right|^2\right) - \mathcal{M}_r\left(|\nabla_T w|^2\right) + k^2\,\mathcal{M}_r\left(|w|^2\right), \tag{31}$$

which we can use for minimizing

$$r^2\left\{\mathcal{M}_r\left(|\nabla u|^2\right) + k^2\,\mathcal{M}_r\left(|u|^2\right)\right\} = \mathcal{M}_r\left(\left|\frac{\partial w}{\partial r}\right|^2\right) + \mathcal{M}_r\left(|\nabla_T w|^2\right) + (k^2 + r^{-2})\,\mathcal{M}_r\left(|w|^2\right) - r^{-1}\,2\,\mathrm{Re}\,\mathcal{M}_r\left(w\frac{\partial w}{\partial r}\right). \tag{32}$$

In fact, let us use the Schwartz inequality in the form of

$$\left|2\,\mathrm{Re}\,\mathcal{M}_r(a\bar{b})\right| \le \frac{1}{2}\mathcal{M}_r\left(|a|^2\right) + 2\,\mathcal{M}_r\left(|b|^2\right) \tag{33}$$

by setting $a = \frac{\partial w}{\partial r}$, $b = r^{-1}w$, which will yield

$$r^2\left\{\mathcal{M}_r\left(|\nabla u|^2\right) + k^2\,\mathcal{M}_r\left(|u|^2\right)\right\} \ge \frac{1}{2}\mathcal{M}_r\left(\left|\frac{\partial w}{\partial r}\right|^2\right) + \mathcal{M}_r\left(|\nabla_T w|^2\right) + (k^2 - r^{-2})\,\mathcal{M}_r\left(|w|^2\right) \ge \frac{1}{2}\left\{F(r) + 3\,\mathcal{M}_r\left(|\nabla_T w|^2\right)\right\} = \frac{1}{2}\left\{F(r) + 3\,\alpha(r)\right\} \tag{34}$$

provided that $k^2 - r^{-2} > \frac{k^2}{2}$. For furnishing the proof, we will attempt to demonstrate that the derivative of $F(r)$ is nonnegative. For this, let us note that $\mathbf{x} = r\,\omega$ and that we can write $\nabla_T w = r^{-1}\frac{\partial w}{\partial \omega}$, such that

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$$F(r) = \mathcal{M}_r\left(\left|\frac{\partial w}{\partial r}\right|^2\right) - r^{-2}\,\mathcal{M}_r\left(\left|\frac{\partial w}{\partial \omega}\right|^2\right) + k^2\,\mathcal{M}_r\left(|w|^2\right), \tag{35}$$

whence

$$\frac{dF}{dr} - 2r^{-3}\,\mathcal{M}_r\left(\left|\frac{\partial w}{\partial \omega}\right|^2\right) = \mathrm{Re}\left\{\mathcal{M}_r\left(\frac{\partial w}{\partial r}\frac{\partial^2 w}{\partial r^2} - r^{-2}\frac{\partial}{\partial r}\left(\frac{\partial w}{\partial \omega}\right)\cdot\frac{\partial w}{\partial \omega} + k^2\frac{\partial w}{\partial r}w\right)\right\} \tag{36}$$

which can be transformed by making use of the fact that u proves the Helmholtz equation and that, consequently, w proves

$$\frac{\partial^2 w}{\partial r^2} + r^{-2}\frac{\partial}{\partial \omega}\cdot\left(\frac{\partial w}{\partial \omega}\right) + k^2 w = 0 \ . \tag{37}$$

Let

$$\frac{dF}{dr} - 2r^{-1}\mathfrak{M}_r\left(|\nabla_r w|^2\right) = -r^{-2}\,\mathrm{Re}\left\{\mathfrak{M}_r\left(\frac{\partial}{\partial r}\left(\frac{\partial w}{\partial \omega}\right)\cdot\frac{\partial w}{\partial \omega} + \frac{\partial w}{\partial r}\frac{\partial}{\partial \omega}\cdot\left(\frac{\partial w}{\partial \omega}\right)\right)\right\}, \tag{38}$$

whose right-hand side is written as

$$-r^2\,\mathrm{Re}\left\{\mathfrak{M}_r\left(\frac{\partial}{\partial \omega}\cdot\left(\frac{\partial w}{\partial r}\frac{\partial w}{\partial \omega}\right)\right)\right\} = 0 \ . \tag{39}$$

In conclusion, we have $\frac{dF}{dr} = \frac{2\alpha(r)}{r} > 0$, which is sufficient to achieve if u is radiant since, in that case, either $u \equiv 0$ or else α will become negligible, resulting in $F > 0$ for a sufficiently large r and for $\liminf_{r\to\infty} F(r) > 0$.

We will now attempt to construct the solution of the diffraction problem by means of a single-layer potential spread over the surface Σ and by means of a volume potential distributed over the interior of Σ. In other words, we will attempt to determine the functions $\mu(\xi)$ and $\tau(\xi)$ such that

$$u(x) = \frac{1}{2\pi}\iint_{\Sigma}\mu(\xi)\frac{e^{-ik|x-\xi|}}{|x-\xi|}\,ds_{\xi} + \frac{1}{4\pi}\iiint_{\mathcal{D}}\tau(\xi)\frac{e^{-ik|x-\xi|}}{|x-\xi|}\,d\xi \tag{40}$$

proves the limit conditions of the problem, in view of the fact that the equation $\Delta u + k^2 u = 0$ is automatically verified outside of Σ. We assume that μ, τ, Σ have all wanted regularity properties for rendering legitimate the operations to be effected next. According to theorem 7, the condition (20) can be written as /II,35

$$\beta\mu(x) + \frac{1}{2\pi}\iint_{\Sigma}\mu(\xi)\left(i\omega\alpha - \beta\frac{d}{dn_x}\right)\frac{e^{-ik|x-\xi|}}{|x-\xi|}\,ds_{\xi} + \tag{41}$$
$$+\frac{1}{4\pi}\iiint_{\mathcal{D}}\tau(\xi)\left(i\omega\alpha - \beta\frac{d}{dn_x}\right)\frac{e^{-ik|x-\xi|}}{|x-\xi|}\,d\xi = -\beta W(x),$$

where the point x is naturally located on Σ. This will yield a single equation for two unknown functions; therefore, let us derive still another equation which we select in such a manner that the resultant equation pair will have only the solution $\mu \equiv 0$, $\tau \equiv 0$ if $w \equiv 0$. This constitutes the reason for having introduced a volume potential. In fact, taking $\tau \equiv 0$, an integral equation in μ is obtained; however, the equation corresponding to $w = 0$ may have a solution in μ which is not identically zero for certain values of k, so that the theorem of the alternative in its simplest form cannot be applied. In fact, for certain values (eigenvalues) of k a value $v_0(x)$ exists which satisfies the following conditions:

$$\begin{cases} v_0(x) \equiv 0 & \text{if } x \text{ is outside of } \mathfrak{D}, \\ \Delta v_0 + k^2 v_0 = 0 & \text{inside of } \mathfrak{D}, \\ v_0 = 0 \;; & \text{on } \Sigma, \text{ inside face,} \end{cases} \tag{42}$$

so that, posing

$$\mu_0(x) = \frac{1}{2}\frac{dv_0}{dn}, \tag{43}$$

where the normal derivative is evaluated from inside of Σ in a direction pointing toward the outside, we will finally have /II,36

$$v_0 = \frac{1}{2\pi}\iint_\Sigma \mu_0(\xi)\,\frac{e^{-ik|x-\xi|}}{|x-\xi|}\,dS_\xi, \tag{44}$$

in the entire space; then, theorem 7, taking into consideration the fact that $v_0 \equiv 0$ at the exterior of Σ, demonstrates:

$$-\beta\mu_0(x) + \frac{1}{2\pi}\iint_\Sigma \mu_0(\xi)\left(-i\omega\alpha + \beta\frac{d}{dn_x}\right)\frac{e^{-ik|x-\xi|}}{|x-\xi|}\,dS_\xi = 0\,. \tag{45}$$

Let us return to eq.(40); we have

$$\Delta u + k^2(u+\tau) = 0, \tag{46}$$

in $\mathfrak{B}$ which suggests adding to eq.(41) the condition $\tau = i\varphi u$, where φ is continuous and zero on Σ, meaning that we can set

$$-\tau(x)+\frac{i\varphi(x)}{2\pi}\iint_{\Sigma}\mu(\xi)\,\frac{e^{-i\kappa|x-\xi|}}{|x-\xi|}\,dS_{\xi}+\frac{i\varphi(x)}{4\pi}\iiint_{\mathfrak{D}}\tau(\xi)\,\frac{e^{-i\kappa|x-\xi|}}{|x-\xi|}\,d\xi=0 \quad (47)$$

for x located within $\mathfrak{D}$; we give the reason for this selection below: Supposing that it exists at all, let us assume a system of functions (μ_0, τ_0), proving eqs.(41) and (47) with $w \equiv 0$; then, the function $v_0(x)$, defined by eq.(40) from μ_0 and τ_0 assumed to be continuous, is defined and continuous in the entire space, proving

$$\begin{cases} \Delta v_0+\kappa^2 v_0=0, & \text{outside of } \mathfrak{D}, \\ \Delta v_0+(\kappa^2+i\varphi)v_0=0, & \text{inside of } \mathfrak{D}. \end{cases} \quad (48)$$

This function is radiant at infinity and proves

$$i\omega\alpha v_0-\beta\frac{dv_0}{dn}=0, \quad \text{on } \Sigma, \text{ exterior} \quad (49)$$

in such a manner that, in virtue of the uniqueness theorem demonstrated above, we will finally have /II,37

$$v_0 \equiv 0, \quad \text{outside of } \mathfrak{D}, \quad (50)$$

and, in virtue of the continuity,

$$v_0=0, \quad \text{on } \Sigma, \text{ interior.} \quad (51)$$

Next, in accordance with the energy identity

$$0=\iint_{\Sigma} v_0\frac{dv_0^*}{dn}\,dS=\iiint_{\mathfrak{D}}\left\{|\nabla v_0|^2-(\kappa^2-i\varphi)|v_0|^2\right\}d\xi, \quad (52)$$

it is obvious that

$$\iiint_{\mathfrak{D}}\varphi\,|v_0|^2\,d\xi, \quad (53)$$

so that, since φ is strictly positive in $\mathfrak{D}$, we will have

$$v_0 \equiv 0, \quad \text{inside of } \mathfrak{D} \quad (54)$$

Thus, v_0 will vanish identically in the entire space, but we will have

$$\begin{cases} 2\mu_0 = \left[\dfrac{dV_0}{dn}\right] , \\ K^2\tau_0 = -(\Delta + K^2)\, V_0 , \end{cases} \tag{55}$$

where $[f] = f_{\Sigma_{ext}} - f_{\Sigma_{int}}$, which results in proving that $\mu_0 \equiv 0$, $\tau_0 \equiv 0$.

The system of integral equations (41) and (47) is a Fredholm system, and it can be demonstrated that, despite the fact that the kernels are not bounded, the alternative is applicable, i.e., that, since the associated homogeneous system does not have a solution, there exists one and only one pair (μ, τ) corresponding to each continuous w. It can also be demonstrated that φ as well as μ and τ are Hölderians. /II,38

It should be noted that the solution obtained by means of (μ, τ) depends on φ while u does not depend on this. This results from the uniqueness theorem either directly or indirectly, as follows: If (μ_1, τ_1) and (μ_2, τ_2) correspond, respectively, to φ_1 and φ_2 and if we pose $\mu^* = \mu_1 - \mu_2$, $\tau^* = \tau_1 - \tau_2$, we have

$$\beta\mu^*(x) + \frac{1}{2\pi}\iint_{\Sigma} \mu^*(\xi)\left(i\omega\alpha - \beta\frac{d}{dn_x}\right)\frac{e^{-ik|x-\xi|}}{|x-\xi|}\,dS_\xi + \\ + \frac{1}{4\pi}\iiint_{\mathfrak{D}} \tau^*(\xi)\left(i\omega\alpha - \beta\frac{d}{dn_x}\right)\frac{e^{-ik|x-\xi|}}{|x-\xi|}\,d\xi = 0, \tag{56}$$

and the uniqueness theorem then demonstrates that u^*, constructed with μ^* and τ^*, vanishes identically.

Let us consider the volume potential

$$U(x) = \frac{1}{4\pi}\iiint_{\mathfrak{D}} \tau(\xi)\,\frac{e^{-ik|x-\xi|}}{|x-\xi|}\,d\xi , \tag{57}$$

and calculate, on Σ, the following values:

$$\begin{cases} \mu_1(x) = -2\,\dfrac{dU}{dn}, & \text{(normal derivative toward the exterior)} \\ \nu_1(x) = U(x) . \end{cases} \tag{58}$$

It is obvious that, outside of Σ, we have

$$U(x) = \frac{1}{2\pi}\iint_{\Sigma} \mu_1(x)\frac{e^{-ik|x-\xi|}}{|x-\xi|}\,dS_\xi + \frac{1}{2\pi}\iint_{\Sigma} \nu_1(\xi)\frac{d}{dn_\xi}\frac{e^{-ik|x-\xi|}}{|x-\xi|}\,dS_\xi, \tag{59}$$

in accordance with theorem 1, so that the solution of the diffraction problem is written in the form of superposition of a single-layer potential and a double-layer potential

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$$u(x) = \frac{1}{2\pi}\iint_{\Sigma}\left(\mu(\xi)+\mu_1(\xi)\right)\frac{e^{-ik|x-\xi|}}{|x-\xi|}\,dS_\xi + \frac{1}{2\pi}\iint_{\Sigma}\nu_1(\xi)\frac{d}{dn_\xi}\frac{e^{-ik|x-\xi|}}{|x-\xi|}\,dS_\xi. \tag{60}$$

The following question is still open. Assuming that $\varphi = \epsilon\phi$, with ϕ being independent of ϵ at $\epsilon \to 0$, will $\mu + \mu_1$ and ν_1 then tend toward the limits and, if so, toward what limits? It seems reasonable to conjecture that, if k does not constitute an eigenvalue for the problem (42), then $\nu_1 \to 0$ but that, if k does constitute an eigenvalue, then ν_1 tends toward a well-defined nonzero limit.

2.6.3 Diffraction by an Obstacle of Small Dimensions

Let a be a scale of a length having the order of magnitude of the maximum diameter of the obstacle; we then wish to solve the problem of diffraction by an approximation method if $ka \ll 1$.

Let $u_0(\mathbf{x})$ be the incident wave and let us limit the calculation to the case in which the obstacle is a rigid body so that, in eq.(20), we must set $\alpha = 0$, $\beta = 1$, $w = -\frac{du_0}{dn}$. It can be demonstrated that the smallest eigenvalue of the problem (42) substantiates

$$k_1 a > \pi \quad , \tag{61}$$

such that, if ka is very small which can be assumed here, it will be possible, in accordance with the conjecture formulated at the end of the preceding Section, to represent the diffracted field in the form of

$$u_d(x) = \frac{1}{2\pi}\iint_{\Sigma}\mu(x)\frac{e^{-ik|x-\xi|}}{|x-\xi|}\,dS_\xi, \tag{62}$$

so that μ must prove

$$\mu(x) - \frac{1}{2\pi} \iint_{\Sigma} \mu(\xi) \frac{d}{dn_x} \frac{e^{-ik|x-\xi|}}{|x-\xi|} \, dS_\xi = \frac{du_0}{dn_x} \tag{63}$$

on Σ.

More accurately, we will assume that the incident wave is a plane wave

$$u_0(x) = e^{-i(K\omega_i \cdot x + \varphi)} , \tag{64}$$

and we will also assume that the origin of the coordinates has been taken somewhere at the interior of the obstacle, for example, at its geometric center. Let us pose

$$x = a\underset{\sim}{x} , \quad \xi = a\underset{\sim}{\xi} , \quad dS_\xi = a^2 \, d\underset{\sim}{S}_{\underset{\sim}{\xi}} , \tag{65}$$

so that the function

$$\underset{\sim}{\mu}(\underset{\sim}{x}) = \mu(a\underset{\sim}{x}) , \tag{66}$$

proves

$$- \underset{\sim}{\mu} + \frac{1}{2\pi} \iint_{\underset{\sim}{\Sigma}} \underset{\sim}{\mu}(\underset{\sim}{\xi}) \frac{d}{d\underset{\sim}{n}_x} \frac{1}{|\underset{\sim}{x} - \underset{\sim}{\xi}|} \, d\underset{\sim}{S}_{\underset{\sim}{\xi}} = iK\, \omega_i \cdot n \, e^{-i(Ka\omega_i \cdot \underset{\sim}{x} + \varphi)}$$
$$+ \frac{Ka}{2\pi} \iint_{\underset{\sim}{\Sigma}} \underset{\sim}{\mu}(\underset{\sim}{\xi}) \frac{d}{d\underset{\sim}{n}_x} \zeta\left(Ka|\underset{\sim}{x} - \underset{\sim}{\xi}|\right) \, d\underset{\sim}{S}_{\underset{\sim}{\xi}} , \tag{67}$$

with

$$\zeta(z) = \frac{1 - e^{-iz}}{z} . \tag{68}$$

The second term on the right-hand side of eq.(67) is at least $O\left((ka)^2\right)$ so that a first approximation (namely, Rayleigh's approximation) can be obtained on substituting eq.(67) by

$$\underset{\sim}{\mu}(\underset{\sim}{x}) - \frac{1}{2\pi}\iint_{\underset{\sim}{\Sigma}} \underset{\sim}{\mu}(\underset{\sim}{\xi}) \frac{d}{dn_{\underset{\sim}{x}}} \frac{1}{|\underset{\sim}{x}-\underset{\sim}{\xi}|} \, dS_{\underset{\sim}{\xi}} = -ika\, \omega_i \cdot n \, e^{-ik\varphi} . \tag{69}$$

Let us consider the equation

$$-f(\underset{\sim}{x}) + \frac{1}{2\pi}\iint_{\underset{\sim}{\Sigma}} f(\underset{\sim}{\xi}) \frac{d}{dn_{\underset{\sim}{x}}} \frac{1}{|\underset{\sim}{x}-\underset{\sim}{\xi}|} \, dS_{\underset{\sim}{\xi}} = \omega_i \cdot n , \tag{70}$$

expressing the function

$$\phi(\underset{\sim}{x}) = \frac{1}{2\pi}\iint_{\underset{\sim}{\Sigma}} f(\underset{\sim}{\xi}) \frac{1}{|\underset{\sim}{x}-\underset{\sim}{\xi}|} \, dS_{\underset{\sim}{\xi}} , \tag{71}$$

which satisfies the Laplace equation and thus can be considered as the velocity potential of the flow of an incompressible fluid, verifying the condition

$$\frac{d\phi}{d\underset{\sim}{n}} = \omega_i \cdot \underset{\sim}{n} \tag{72}$$

along the wall $\underset{\sim}{\Sigma}$ of the obstacle. This means that ϕ is the velocity potential of flow of an incompressible fluid, produced by the displacement of the obstacle at a velocity ω.

Problem 10: Assuming that the obstacle is a sphere of radius 1, prove that

$$\phi_e(\underset{\sim}{x}) = -\frac{\omega_i \cdot \underset{\sim}{x}}{2|\underset{\sim}{x}|^3} \tag{73}$$

satisfies the conditions previously imposed on ϕ, and demonstrate that

$$\phi_i(\underset{\sim}{x}) = -\frac{1}{2}\,\omega_i \cdot \underset{\sim}{x} \tag{74}$$

proves the Laplace equation at the interior of Σ and, on this same Σ, assumes the same values as ϕ_e. Derive from this that, in the case of the sphere with radius 1, the solution of eq.(70) is given by

$$f(x) = -\frac{3}{4}\,\frac{\omega_i \cdot x}{|x|}. \tag{75}$$

Problem 11: Considering the ellipsoid

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} + \frac{z^2}{\gamma^2} = 1\,, \tag{76}$$

we pose

$$\frac{x^2}{\alpha^2+\theta} + \frac{y^2}{\beta^2+\theta} + \frac{z^2}{\gamma^2+\theta} - 1 = \frac{(\lambda-\theta)(\mu-\theta)(\nu-\theta)}{(\alpha^2+\theta)(\beta^2+\theta)(\gamma^2+\theta)}\,. \tag{77}$$

Derive from this that

$$\left\{\begin{aligned} x^2 &= \frac{(\alpha^2+\lambda)(\alpha^2+\mu)(\alpha^2+\nu)}{(\alpha^2-\beta^2)(\alpha^2-\gamma^2)}\,, \\ y^2 &= \frac{(\beta^2+\lambda)(\beta^2+\mu)(\gamma^2+\nu)}{(\beta^2-\gamma^2)(\beta^2-\alpha^2)}\,, \\ z^2 &= \frac{(\gamma^2+\lambda)(\gamma^2+\mu)(\gamma^2+\nu)}{(\gamma^2-\alpha^2)(\gamma^2-\beta^2)}\,, \\ \frac{x^2}{\alpha^2+\lambda} + \frac{y^2}{\beta^2+\lambda} + \frac{z^2}{\gamma^2+\lambda} &= \frac{(\lambda-\mu)(\lambda-\nu)}{(\alpha^2+\lambda)(\beta^2+\lambda)(\gamma^2+\lambda)}\,, \end{aligned}\right. \tag{78}$$

and prove the relation

$$\begin{aligned} (dx)^2 + (dy)^2 + (dz)^2 = 4\Big\{ &\frac{(\alpha^2+\lambda)(\beta^2+\lambda)(\gamma^2+\lambda)}{(\lambda-\mu)(\lambda-\nu)}\,(d\lambda)^2 + \\ &+ \frac{(\alpha^2+\mu)(\beta^2+\mu)(\gamma^2+\mu)}{(\mu-\lambda)(\mu-\nu)}\,(d\mu)^2 + \frac{(\alpha^2+\nu)(\beta^2+\nu)(\gamma^2+\nu)}{(\nu-\lambda)(\nu-\mu)}\,(d\nu)^2\Big\} \end{aligned} \tag{79}$$

Problem 12: Demonstrate that, if

$$(dx)^2 + (dy)^2 + (dz)^2 = (h_1\,dx_1)^2 + (h_2\,dx_2)^2 + (h_3\,dx_3)^2, \qquad (80)$$

we will automatically have /II,43

$$\Delta \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \equiv h_1 h_2 h_3 \left\{ \frac{\partial}{\partial x_1}\left(\frac{h_1}{h_2 h_3}\frac{\partial}{\partial x_1}\right) + \frac{\partial}{\partial x_2}\left(\frac{h_2}{h_3 h_1}\frac{\partial}{\partial x_2}\right) + \frac{\partial}{\partial x_3}\left(\frac{h_3}{h_1 h_2}\frac{\partial}{\partial x_3}\right)\right\}. \qquad (81)$$

Problem 13: Resuming the notations of Problem 11 and posing

$$\delta = \left\{ (\alpha^2+\lambda)(\beta^2+\lambda)(\gamma^2+\lambda)\right\}^{\frac{1}{2}}, \qquad (82)$$

demonstrate that the expression

$$\phi = \int_\lambda^\infty \frac{d\lambda_1}{\delta(\lambda_1)}, \qquad (83)$$

is a solution of the Laplace equation. Posing

$$A = \alpha\beta\gamma \int_0^\infty \frac{d\lambda}{(\alpha^2+\lambda)\,\delta}, \quad B = \alpha\beta\gamma \int_0^\infty \frac{d\lambda}{(\beta^2+\lambda)\,\delta}, \quad C = \alpha\beta\gamma \int_0^\infty \frac{d\lambda}{(\gamma^2+\lambda)\,\delta}, \qquad (84)$$

demonstrate that

$$\phi_a = -\frac{\alpha\beta\gamma}{2-A}\, x\, \omega_i \cdot i \int_\lambda^\infty \frac{d\lambda_1}{(\alpha^2+\lambda_1)\,\delta(\lambda_1)} - \frac{\alpha\beta\gamma}{2-B}\, y\, \omega_i \cdot j \int_\lambda^\infty \frac{d\lambda_1}{(\beta^2+\lambda_1)\,\delta(\lambda_1)} - \frac{\alpha\beta\gamma}{2-C}\, z\, \omega_i \cdot k \int_\lambda^\infty \frac{d\lambda_1}{(\gamma^2+\lambda_1)\,\delta(\lambda_1)} \qquad (85)$$

is a solution of the Laplace equation at the exterior of the ellipsoid $\lambda = 0$. Here, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors of the axes $O\,\underset{\sim}{x}, \underset{\sim}{y}, \underset{\sim}{z}$. Demonstrate that, on the ellipsoid $\lambda = 0$, we have

$$\frac{d\phi_e}{dn} = \omega_i \cdot n. \tag{86}$$

Show that the expression

$$\phi_i = -\frac{A}{2-A}\,\omega_i \cdot i\,\underset{\sim}{x} - \frac{B}{2-B}\,\omega_i \cdot j\,\underset{\sim}{y} - \frac{C}{2-C}\,\omega_i \cdot k\,\underset{\sim}{z} \tag{87}$$

satisfies the Laplace equation at the interior of the ellipsoid $\lambda = 0$ and, /II,44 on this latter, assumes the same values as φ_e; derive from this that, in the case of the ellipsoid (76), the solution of eq.(70) is written as follows:

$$f(\underset{\sim}{x}) = \omega_i \cdot \left\{\frac{2}{2-A}\,ii + \frac{2}{2-B}\,jj + \frac{2}{2-C}\,kk\right\} \cdot n. \tag{88}$$

Pick up the result of Problem 11, as a particular case.

Let us return to eq.(67) by passing over the intermediary of

$$-f(\underset{\sim}{x}) + \frac{1}{2\pi}\iint_{\underset{\sim}{\Sigma}} f(\underset{\sim}{\xi})\,\frac{d}{dn_x}\,\frac{1}{|\underset{\sim}{x}-\underset{\sim}{\xi}|}\,d\underset{\sim}{S}_{\xi} = g(\underset{\sim}{x}), \tag{89}$$

whose unique solution is expressed in the form of

$$f(\underset{\sim}{x}) = \iint_{\underset{\sim}{\Sigma}} g(\underset{\sim}{\xi})\,\Gamma(\underset{\sim}{x}, \underset{\sim}{\xi})\,d\underset{\sim}{S}_{\underset{\sim}{\xi}} \tag{90}$$

From now on, we can write eq.(67) in the form

$$\begin{aligned}\underset{\sim}{\mu}(\underset{\sim}{x}) - (ka)^2 \iint_{\underset{\sim}{\Sigma}} \underset{\sim}{\mu}(\underset{\sim}{\xi})\,\Lambda(\underset{\sim}{x}, \underset{\sim}{\xi}, ka)\,d\underset{\sim}{S}_{\xi} \\ = ika\,e^{-i\varphi} \iint_{\underset{\sim}{\Sigma}} \omega_i \cdot n(\xi)\,e^{-ika\,\omega_i \cdot \underset{\sim}{\xi}}\,\Gamma(\underset{\sim}{x}, \underset{\sim}{\xi})\,d\underset{\sim}{S}_{\underset{\sim}{\xi}}\end{aligned} \tag{91}$$

with

$$\Lambda(\underset{\sim}{x},\underset{\sim}{\xi},ka) = \frac{1}{2\pi}\iint_{\underset{\sim}{\Sigma}} \Gamma(\underset{\sim}{x},\underset{\sim}{\eta})\,\frac{d}{dn_{\xi}}\left(\frac{\zeta(ka|\xi-\eta|)}{ka}\right) dS_{\underset{\sim}{\eta}} \tag{92}$$

and demonstrate that Λ can be expanded in an integral power series of ka

$$\Lambda = \sum_{n=0}^{\infty} (ka)^n \Lambda_n(\underset{\sim}{x},\underset{\sim}{\xi}) \,. \tag{93}$$

A study of eq.(92) will indicate clearly that we wish to define $\underset{\sim}{\mu}(\underset{\sim}{x})$ in the form of an expansion in series /II,45

$$e^{i\varphi}\underset{\sim}{\mu}(\underset{\sim}{x}) = \sum_{n=1}^{\infty} (ka)^n \underset{\sim}{\mu}_n(\underset{\sim}{x}), \tag{94}$$

where the quantities $\underset{\sim}{\mu}_n(\mathbf{x})$ are readily obtained by substitution. It is certain that the series (94) converges for a sufficiently small ka, i.e., that this series has a nonzero convergence range. This is easily shown by starting from

$$\underset{\sim}{\mu}^{\lambda}(\underset{\sim}{x}) - \lambda\iint_{\underset{\sim}{\Sigma}} \underset{\sim}{\mu}^{\lambda}(\underset{\sim}{\xi})\,\Lambda(\underset{\sim}{x},\underset{\sim}{\xi};ka)\, dS_{\underset{\sim}{\xi}} = g(\underset{\sim}{x}), \tag{95}$$

whose solution can be expressed in the form of a power series in λ, converging for $|\lambda| < \lambda_0$. Thus,

$$\underset{\sim}{\mu}^{\lambda}(\underset{\sim}{x}) = g(x) + \sum_{n=1}^{\infty} \lambda^n \iint_{\underset{\sim}{\Sigma}} g(\underset{\sim}{\xi})\,\overset{(n)}{\Lambda}(x,\xi;ka)\, dS_{\xi}, \tag{96}$$

with

$$\begin{cases} \overset{(1)}{\Lambda} = \Lambda, \\ \overset{(n+1)}{\Lambda}(\underset{\sim}{x};\underset{\sim}{\xi};ka) = \iint_{\underset{\sim}{\Sigma}} \Lambda(\underset{\sim}{x},\underset{\sim}{\eta};ka)\,\overset{(n)}{\Lambda}(\underset{\sim}{\eta},\underset{\sim}{\xi};ka)\, dS_{\underset{\sim}{\eta}} \end{cases} \tag{97}$$

which shows, by recurrence, that $\overset{(n)}{\Lambda}$ can be expanded in integral power series of ka, i.e.,

$$\overset{(n)}{\Lambda}(\underset{\sim}{x}, \underset{\sim}{\xi}; \kappa a) = \sum_{p=0}^{\infty} (\kappa a)^p \overset{(n)}{\Lambda}_p(\underset{\sim}{x}, \underset{\sim}{\xi}), \tag{97}$$

such that

$$\overset{\lambda}{\underset{\sim}{\mu}}(\underset{\sim}{x}) = g(\underset{\sim}{x}) + \sum_{n=1}^{\infty} \lambda^n \sum_{p=0}^{\infty} (\kappa a)^p \iint_{\underset{\sim}{\Sigma}} g(\underset{\sim}{\xi}) \overset{(n)}{\Lambda}_p(\underset{\sim}{x}, \underset{\sim}{\xi}) \, dS_{\underset{\sim}{\xi}} . \tag{98}$$

In this expression, it is convenient to set

/II,46

$$g(\underset{\sim}{x}) = e^{-i\varphi} \sum_{n=0}^{\infty} \frac{(-i\kappa a)^{n+1}}{n!} \iint_{\underset{\sim}{\Sigma}} (\omega_i \cdot \underset{\sim}{n}(\underset{\sim}{\xi}) \, ((\omega_i \cdot \underset{\sim}{\xi})^n \, \Gamma(\underset{\sim}{x}, \underset{\sim}{\xi}) \, dS_{\underset{\sim}{\xi}} \tag{99}$$

and to replace λ by $(ka)^2 < |\lambda_0|$. By rearrangement, we obtain eq.(94) and leave to the reader the task of writing $\underset{\sim}{\mu_r}(\underset{\sim}{x})$. It is not impossible that the range of convergence of eq.(94) is exactly equal to $k_1 a$ where k_1 is the first eigenvalue corresponding to the first natural mode of acoustic vibration at the interior of $\underset{\sim}{\Sigma}$; however, this cannot be confirmed with absolute certainty at present.

Theorem 12: Let, in reduced variables $\underset{\sim}{x}$,

$$\phi_1(\underset{\sim}{x}) = \frac{1}{2\pi} \iint_{\underset{\sim}{\Sigma}} \frac{f_1(\underset{\sim}{x})}{|\underset{\sim}{x} - \underset{\sim}{\xi}|} \, dS_{\underset{\sim}{\xi}} \tag{100}$$

be the velocity potential of a flow of incompressible fluid, produced by the motion of the obstacle $\underset{\sim}{\Sigma}$ at - unit - velocity ω_1 ; let, similarly,

$$\phi_2(\underset{\sim}{x}) = \frac{1}{2\pi} \iint_{\underset{\sim}{\Sigma}} \frac{f_2(\underset{\sim}{\xi})}{|\underset{\sim}{x} - \underset{\sim}{\xi}|} \, dS_{\underset{\sim}{\xi}} \tag{101}$$

be the velocity potential of a flow of incompressible fluid produced by an expansion of the obstacle in the direction ω_1 - two points of the obstacle placed

on a parallel to ω_1 travel away from each other with a speed equal to their reduced distance, i.e., $\frac{d\phi_2}{dn} = (\omega_1 \cdot n)(\omega_1 \cdot x)$; in this case, the solution of eq.(67) is written as follows:

$$\mu(x) = ika\, f_1(x) + (ka)^2 f_2(x) + O((ka)^3). \tag{102}$$

Returning to the diffracted field $u_d(x)$, expressed by eq.(62), passing to reduced variables, and making use of eq.(102), we will obtain /II,47

$$u_d(x) = \frac{ik^2a^2}{2\pi} \iint_{\Sigma} \left\{ f_1(\xi) - ika\, f_2(\xi) + O((ka)^2) \right\} \frac{e^{-i|kx - ka\xi|}}{|kx - ka\xi|}\, dS_\xi \tag{103}$$

which leads to the expression

$$\frac{e^{-i|kx - ka\xi|}}{|kx - ka\xi|} = \frac{e^{-ik|x|}}{k|x|}\left(1 + ika\,\omega\cdot\xi\right) + ka\,\omega\cdot\xi\,\frac{e^{-ik|x|}}{k^2|x|^2} + O((ka)^2), \tag{104}$$

if we pose

$$x = |x|\,\omega \tag{105}$$

in such a manner that the following is obtained:

$$u_d(x) = \frac{ik^2a^2}{2\pi}\frac{e^{-ik|x|}}{k|x|}\iint_{\Sigma}\left\{f_1(\xi) + ika\left[\omega\cdot\xi\, f_1(\xi) - f_2(\xi)\right]\right\} dS_\xi + \frac{ik^3a^3}{2\pi}\frac{e^{-ik|x|}}{k^2|x|^2}\iint_{\Sigma}\omega\cdot\xi\, f_1(\xi)\, dS_\xi + O(k^5a^5). \tag{106}$$

Let us now attempt to interpret the integrals on the right-hand side of the

equation. It is obvious, primarily, that we have

$$\iint_{\underset{\sim}{\Sigma}} f_1(\underset{\sim}{\xi})\, d\underset{\sim}{S}_{\underset{\sim}{\xi}} = 0 . \tag{107}$$

Problem 14: Demonstrate that the mass flux of the velocity field, obtained from the potential

$$\phi(x) = \frac{1}{2\pi} \iint_{\underset{\sim}{\Sigma}} f(\xi) \frac{d\underset{\sim}{S}_{\xi}}{|\underset{\sim}{x} - \underset{\sim}{\xi}|} \tag{108}$$

across a surface $\underset{\sim}{S}$ surrounding $\underset{\sim}{\Sigma}$, is equal to $-2 \iint_{\underset{\sim}{\Sigma}} f(\underset{\sim}{\xi})\, d\underset{\sim}{S}_{\underset{\sim}{\xi}}$. Derive from /II,48 this the relation (107).

According to the Problem 14, we have

$$2 \iint_{\underset{\sim}{\Sigma}} f_2(\underset{\sim}{\xi})\, d\underset{\sim}{S}_{\underset{\sim}{\xi}} = -\iint_{\underset{\sim}{\Sigma}} (\omega_i \cdot n(\xi)) (\omega_i \cdot \xi)\, d\underset{\sim}{S}_{\xi} = -\underset{\sim}{\mathfrak{V}}, \tag{109}$$

denoting by $\underset{\sim}{\mathfrak{V}}$ the (reduced) volume of the obstacle; all that remains is to calculate $\iint_{\underset{\sim}{\Sigma}} \omega \cdot \underset{\sim}{\xi}\, f_1(\underset{\sim}{\xi})\, d\underset{\sim}{S}_{\underset{\sim}{\xi}}$. Let us note that $f_1(\xi)$ depends linearly on ω_i, i.e.,

$$f_1(\underset{\sim}{\xi}) = \mathbb{F}_1(\xi) \cdot \omega_i , \tag{110}$$

where $\mathbb{F}_1$ denotes a vectorial function. This leads to forming the tensor

$$\dot{\underset{\sim}{T}} = -\iint_{\underset{\sim}{\Sigma}} \underset{\sim}{\xi}\, \mathbb{F}_1(\underset{\sim}{\xi})\, d\underset{\sim}{S}_{\xi} , \tag{111}$$

by means of which we obtain

$$\iint_{\underset{\sim}{\Sigma}} \omega \cdot \xi\, f_1(\xi)\, d\underset{\sim}{S}_{\underset{\sim}{\xi}} = -\omega \cdot \underset{\sim}{T} \cdot \omega_i . \tag{112}$$

The tensor $\mathbb{T}$ is interpreted in terms of the kinetic energy of the flow of an incompressible fluid, represented by ϕ_1. In fact, let us assume that $\underset{\sim}{\Sigma}$ is filled with the same fluid and that it thus entrains a motion whose velocity potential is $\phi^{(i)} = \omega_1 \cdot \xi$, while $\phi^{(e)} = \phi_1 = \frac{1}{2n} \iint\limits_{\underset{\sim}{\Sigma}} \frac{f_1(\xi)}{|x - \xi|}\, d\underset{\sim}{S}_{\xi}$ describes the motion of the fluid at the exterior. On the surface of $\underset{\sim}{\Sigma}$, we have $\phi^{(i)} \neq \phi^{(e)}$; /II,49
thus, let us pose

$$[\phi] = \phi^{(e)} - \phi^{(i)}. \tag{113}$$

<u>Problem 15</u>: The kinetic energy of a flow of incompressible fluid, i.e.,

$$\zeta = \frac{1}{2} \iiint\limits_{V} |\nabla \phi|^2\, dv, \tag{114}$$

is given by

$$\zeta = -\frac{1}{2} \iint\limits_{S} \phi \frac{d\phi}{dn}\, ds, \tag{115}$$

where S is the surface limiting the domain $\mathfrak{V}$. This formula is valid if $\mathfrak{V}$ contains the point at infinity, provided that ϕ vanishes there. The normal derivative is evaluated in the direction pointing toward $\mathfrak{V}$. The surface S is an integer at a finite distance.

Let us apply the result of Problem 15 to the motion at the exterior of $\underset{\sim}{\Sigma}$, and note

$$\underset{\sim}{\zeta} = \underset{\sim}{\zeta}^{(e)} + \underset{\sim}{\zeta}^{(i)}, \tag{116}$$

which is the kinetic energy; this will yield

$$\underset{\sim}{\zeta} = -\frac{1}{2} \iint\limits_{\underset{\sim}{\Sigma}} [\phi]\, \omega_i \cdot \underset{\sim}{n}\, d\underset{\sim}{S} = -\frac{1}{2} \iint\limits_{\underset{\sim}{\Sigma}} \phi^{(e)}\, \omega_i \cdot \underset{\sim}{n}\, d\underset{\sim}{S} + \frac{1}{2} \iint\limits_{\underset{\sim}{\Sigma}} (\omega_i \cdot \underset{\sim}{\xi})(\omega_i \cdot \underset{\sim}{n})\, d\underset{\sim}{S}. \tag{117}$$

Now, at a point exterior to $\underset{\sim}{\Sigma}$, we have

$$\phi(\underset{\sim}{x}) = \frac{1}{4\pi} \iint_{\underset{\sim}{\Sigma}} [\phi] \frac{d}{dn_{\underset{\sim}{\xi}}} \frac{1}{|\underset{\sim}{x} - \underset{\sim}{\xi}|} \, dS_{\underset{\sim}{\xi}} \, . \tag{118}$$

Problem 16: Establish eq.(118) by starting from /II,50

$$\left\{ \begin{aligned} \phi^{(e)} &= \frac{1}{4\pi} \iint_{\underset{\sim}{\Sigma}} \left\{ \phi^{(e)} \frac{d}{dn_{\underset{\sim}{\xi}}} \frac{1}{|\underset{\sim}{x} - \underset{\sim}{\xi}|} - \frac{d\phi^{(e)}}{dn_{\underset{\sim}{\xi}}} \frac{1}{|\underset{\sim}{x} - \underset{\sim}{\xi}|} \right\} dS_{\underset{\sim}{\xi}} \, , \\ 0 &= \frac{1}{4\pi} \iint_{\underset{\sim}{\Sigma}} \left\{ \phi^{(i)} \frac{d}{dn_{\underset{\sim}{\xi}}} \frac{1}{|\underset{\sim}{x} - \underset{\sim}{\xi}|} - \frac{d\phi^{(i)}}{dn_{\underset{\sim}{\xi}}} \frac{1}{|\underset{\sim}{x} - \underset{\sim}{\xi}|} \right\} dS_{\underset{\sim}{\xi}} \, . \end{aligned} \right. \tag{119}$$

Since, for $|\underset{\sim}{x}| \to \infty$, we have $\dfrac{d}{dn_{\underset{\sim}{\xi}}} \dfrac{1}{|\underset{\sim}{x} - \underset{\sim}{\xi}|} = \dfrac{\underset{\sim}{x} \cdot \underset{\sim}{n}}{|\underset{\sim}{x}|^3}$, then the asymptotic behavior of ϕ_1, for large values of $|\underset{\sim}{x}|$ is given by

$$\phi_1(\underset{\sim}{x}) \cong \frac{\underset{\sim}{x}}{4\pi |\underset{\sim}{x}|^3} \iint_{\underset{\sim}{\Sigma}} [\phi] \, \underset{\sim}{n} \, dS_{\underset{\sim}{\xi}} = \frac{1}{4\pi |\underset{\sim}{x}|^2} \iint_{\underset{\sim}{\Sigma}} [\phi] \, \omega \cdot \underset{\sim}{n} \, dS_{\underset{\sim}{\xi}} \, , \tag{120}$$

so that, according to eqs.(110) and (112), we have

$$\phi_1(\underset{\sim}{x}) \cong - \frac{\omega \cdot \underset{\sim}{\mathbb{T}} \cdot \omega_i}{2\pi |\underset{\sim}{x}|^2} \, , \tag{121}$$

from which the formula

$$\underset{\sim}{\zeta} = \omega_i \cdot \underset{\sim}{\mathbb{T}} \cdot \omega_i = \underset{\sim}{\zeta}^{(e)} + \frac{1}{2} \underset{\sim}{\nu} \, , \tag{122}$$

is obtained which, introducing the identity tensor II, can be written as follows:

$$\omega_i \cdot \left(\mathbb{T} - \frac{1}{2} \underset{\sim}{\mathcal{V}}\, \mathbb{I}\right) \cdot \omega_i = \underset{\sim}{\mathfrak{S}}^{(e)}(\omega_i) \ , \tag{123}$$

which completely defines $\mathbb{T}$, since it is obvious that $\underset{\sim}{\mathfrak{S}}^{(e)}$ is a quadratic form of ω_i and that $\mathbf{T}$ is a symmetric tensor. With respect to this latter point, let us note that, if $\phi^{(e)}(\omega)$ is the rotation indicating that we have here a /II,51 velocity potential corresponding to a translation of the obstacle at unit velocity in the direction ω, i.e., $\frac{d\phi}{d\underset{\sim}{n}}\Big|_{\underset{\sim}{\Sigma}} = \omega \cdot n$, we will have

$$\omega_1 \cdot \mathbb{T} \cdot \omega_2 - \omega_2 \cdot \mathbb{T} \cdot \omega_1 = \frac{1}{2} \iint\limits_{\underset{\sim}{\Sigma}} \left\{ [\phi](\omega_1) \frac{d\phi(\omega_2)}{d\underset{\sim}{n}} - [\phi](\omega_2) \frac{d\phi(\omega_1)}{d\underset{\sim}{n}} \right\} d\underset{\sim}{S} = 0 \ . \tag{124}$$

Theorem 14: Let $a^3 \underset{\sim}{\mathfrak{B}}$ be the volume of the obstacle and let

$$\mathfrak{S}^{(e)} = \rho_0 U^2 a^3 \omega \cdot \left(\underset{\sim}{\mathbb{T}} - \frac{1}{2} \underset{\sim}{\mathcal{V}}\, \mathbb{I}\right) \cdot \omega \tag{125}$$

be the kinetic energy of the steady-state flow at the exterior of the obstacle of an incompressible fluid produced by the displacement of the obstacle with a velocity U in the direction ω, whereas, if a sound wave

$$u_0 = e^{-ik\, x \cdot \omega_i} \tag{126}$$

is incident on the obstacle, the resultant diffracted sound wave, in Rayleigh's approximation $ka \ll 1$, is given by the formula ($x = |x|\, \omega$)

$$u_d(x) = \frac{k^3 a^3}{2\pi} \frac{e^{-ik|x|}}{k|x|} \left\{ \omega \cdot \underset{\sim}{\mathbb{T}} \cdot \omega_i + \frac{1}{2} \underset{\sim}{\mathcal{V}} \right\} + \\ + \frac{ik^3 a^3}{2\pi} \frac{e^{-ik|x|}}{k^2|x|^2} \omega \cdot \underset{\sim}{\mathbb{T}} \cdot \omega_i + O(k^5 a^5) \ . \tag{127}$$

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CHAPTER III /III,1

RESOLUTION OF SOME DIFFRACTION PROBLEMS

3.1 Diffraction by an Aperture in a Plane Screen

3.1.1 General Principles

The plane z = 0 is assumed as occupied by a rigid thin screen E, pierced by an aperture O. At infinity, in the half-space z < 0, the sound field is supposed to contain, among others, a plane wave

$$u_i(x) = e^{-ik\,\omega\cdot x} \qquad \omega\cdot\vec{z} > 0, \tag{1}$$

where $\vec{z}$ denotes the unit vector of the oz axis. Speaking more generally, it could be assumed that

$$u_i(x) = \iint_{\omega_z > 0} F(\omega)\, e^{-ik\,\omega\cdot x}\, d\omega . \tag{2}$$

We propose to find a sound field $u_0 + u$ whose normal velocity component vanishes on E and which proves the Helmholtz equation everywhere outside of E which, naturally, is continuously differentiable across O, satisfying the conditions to be prescribed at an infinite distance from the center of O.

Let us define the total field, in the form of

$$u(x) = u_i(x) + u_r(x) + u_d(x) , \tag{3}$$

where the reflected field u_r is such that $u_i + u_r$ will yield the total field in the absence of any aperture. We then obviously have /III,2

$$u_r(x, y, z) = \pm\, u_i(x, y \mp z) \qquad \begin{matrix} +, - & \leftrightarrow & z < 0 \\ -, + & \leftrightarrow & z > 0 \end{matrix} \tag{4}$$

since $u_i + u_r$ satisfy all conditions of the problem, provided that E occupies the entire plane z = 0.

The quantity u_d is to represent the field diffracted by the aperture and satisfying

$$\begin{cases} \Delta u_d + k^2 u_d = 0 & \text{outside of E} \\ \dfrac{\partial u_d}{\partial z} = 0 & \text{on E} \\ [u_d] = 2u_i & \text{on 0} \quad ([f] = f_{z=+0} - f_{z=-0}) . \end{cases} \tag{5}$$

However, these conditions are not sufficient for unequivocally determining u_d, and require the addition of conditions at infinity and of conditions in the vicinity of the boundary of the aperture.

Problem 1: Let us assume that u verifies $\Delta u + k^2 u = 0$ in the neighborhood of infinity, excluding the plane z = 0 on traversing of which u undergoes a discontinuity [u], with $\frac{\partial u}{\partial z}$ being continuous. Then, establish the relation

$$\frac{\partial^2}{\partial r^2}\left(r \mathfrak{M}_r^{x}(u)\right) + k^2 r \mathfrak{M}_r^{x}(u) + \frac{z}{2r} \frac{\partial}{\partial r}\left\{ {}^{P}\mathfrak{M}^{x_p}_{\sqrt{r^2-z^2}}([u]) \right\} = 0 \tag{6}$$

with the following notations: x_p is the projection of the point x onto the plane z = 0 while ${}^{P}\mathfrak{M}^{y}_{\rho}$ is the mean on the circle with center y and radius ρ of the plane z = 0. In addition, [u] = u(z = +0) - u(z = -0).

Problem 2: For $\mathfrak{M}_r^{x}(u)$, establish the representation formula /III,3

$$\mathfrak{M}_r^{x}(u) = u_I'(x) \frac{e^{ikr}}{2ikr} + u_{II}'(x) \frac{e^{-ikr}}{2ikr} - \int_{r_0}^{r} \frac{z \sin[k(r-\rho)]}{2kr} \frac{\partial}{\partial \rho}\left\{ {}^{P}\mathfrak{M}^{x_P}_{\sqrt{\rho^2-z^2}}([u]) \right\} d\rho . \tag{7}$$

According to Problems 1 and 2, it seems reasonable to assume that the asymptotic behavior of $\mathfrak{M}_r^{x}(u)$ at $r \to \infty$ is still of the same type as if there were no screen. Consequently, we can impose the following condition on u_d:

Radiation condition:

$$\lim_{r \to \infty} \left\{ r \left(\frac{\partial u_d}{\partial r} + ik u_d \right) \right\} = 0 . \tag{8}$$

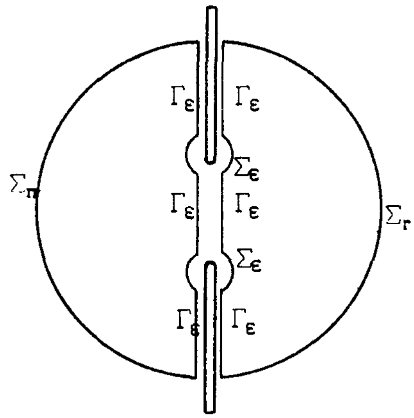
It then remains to define the behavior of u_d in the vicinity of the aperture rim. It is expected that u_d is in $\epsilon^{1/2}$ and ∇u_d in $\epsilon^{-1/2}$, if ϵ denotes the distance from the rim. In fact, it is sufficient to impose a much weaker condition which, for the edge constituting the boundary, represents a

Non-radiation condition:

$$\lim_{\varepsilon \to 0} \varepsilon \, \mathfrak{M}^{x_0}_{\Gamma_\varepsilon(x_0)} \left(|\nabla u_d| \right) = 0, \tag{9}$$

where $\Gamma_\varepsilon(x_0)$ is a circle of radius ϵ with its axis along the tangent in x_0 to the boundary and where $\mathfrak{M}^{x_0}_{\Gamma_\varepsilon}$ denotes the mean on this circle.

The mathematical problem consists now in determining u_d, satisfying the conditions (4), (8), (9). Let us demonstrate that the solution, if it /III,4



exists at all, is quite unique. Let us apply the energy identity to the difference v of the two solutions, by using $\Sigma = \Sigma_r + \Sigma_\varepsilon + \Gamma_\varepsilon$, as indicated on the accompanying diagram. Since, on Γ_ε, $[v]$ or $\frac{\partial v}{\partial z}$ vanishes, we will have the following expression in virtue of the condition (9):

$$\lim_{r \to \infty} \iint_{\Sigma_r} \left(v^* \frac{\partial v}{\partial r} - v \frac{\partial v^*}{\partial r} \right) ds = 0, \tag{10}$$

after having made $\epsilon \to 0$. Now, the condition (8) demonstrates that the following becomes valid:

$$u(r,\omega) = A_0(\omega)\frac{e^{-ikr}}{r} - \int_r^{\infty} \frac{e^{-ik(r-\rho)}}{\rho} A(\rho,\omega)\, d\rho, \tag{11}$$

where the function $A(\rho, \omega)$ tends to zero as soon as $\rho \to \infty$, while eq.(10) shows that

$$A_0(\omega) \equiv 0 . \tag{12}$$

If the condition (8) is made slightly more rigorous by stipulating not only that $A(r, \omega)$ tends to zero but also that its gradient in ω tends to zero, we will again obtain the relation (2.29) of Chapter II, which is achieved because of theorems 10 and 11 of Chapter II. For proving theorem 10, it will be noted that the discontinuity of v on z = 0 enters only eq.(2.39) of Chapter II and that $\frac{\partial v}{\partial \omega}$ vanishes on z = 0. Let us also state that there is no rigorous assurance for the convergence of the integral in eq.(11) because of the only condition (8). /III,5

We thus have (demonstrated) the uniqueness of u_d, satisfying the conditions (4), (8), (9), from which it results that we necessarily must have

$$u_d(x,y,z) + u_d(x,y,-z) = 0 . \tag{13}$$

Let us now consider the problem of diffraction, corresponding to the case in which u vanishes on the screen, and let us note

$$u^{II} = u_i + u_r^{II} + u_d^{II}, \tag{14}$$

which thus constitutes the solution since the solution of the preceding problem is

$$u^{I} = u_i + u_r^{I} + u_d^{I} . \tag{15a}$$

Obviously, we have

$$u_r^{II}(x,y,z) = -u_i(x,y,\mp z) \qquad \begin{matrix} - \leftrightarrow z<0 \\ + \leftrightarrow z>0, \end{matrix} \tag{15b}$$

so that u_d must prove

$$
\begin{cases}
\Delta u_d^{II} + k^2 u_d^{II} = 0 , & \\
u_d^{II} = 0 , & \text{at } \mathcal{E} \\
\left[\dfrac{\partial u_d^{II}}{\partial z}\right] = 2\,\dfrac{\partial u_i}{\partial z} , & \text{at } \sigma
\end{cases}
\tag{16}
$$

again using eqs.(8) and (9). It is obvious that we here have

$$
u_d^{II}(x,y,-z) - u_d^{II}(x,y,z) = 0 . \tag{17}
$$

Let us now interchange screen and aperture, which yields two problems that are complementary to the preceding problems and whose solutions are as follows: /III,6

$$
\begin{cases}
{}^c u^{I} = u_i + u_r^{I} + {}^c u_d^{I} , \\
{}^c u^{II} = u_i + u_r^{II} + {}^c u_d^{II} ,
\end{cases}
\tag{18}
$$

with the conditions

$$
\begin{cases}
\dfrac{\partial\, {}^c u_d^{I}}{\partial z} = 0 , & \text{at } \sigma \\
\left[{}^c u_d^{I}\right] = 2 u_i , & \text{at } \mathcal{E}
\end{cases}
\tag{19}
$$

$$
\begin{cases}
{}^c u_d^{II} = 0 , & \text{at } \sigma \\
\left[\dfrac{\partial\, {}^c u_d^{II}}{\partial z}\right] = 2 u_i , & \text{at } \mathcal{E}
\end{cases}
\tag{20}
$$

and, naturally, with the Helmholtz equation and the conditions (8) and (9). This time, we have

$$
\begin{cases}
{}^c u_d^{I}(x,y,-z) + {}^c u_d^{I}(x,y,z) = 0 , \\
{}^c u_d^{II}(x,y,-z) - {}^c u_d^{II}(x,y,z) = 0 .
\end{cases}
\tag{21}
$$

Babinet's principle: If we have the solution of two of four problems of diffraction, we will automatically have the solution of the two other problems, with the formulas

$$\left\{\begin{array}{ll} u_d^{I} \pm {}^{c}u_d^{II} = \pm \cdot u_i, & + \leftrightarrow z>0 \\ u_d^{II} + {}^{c}u_d^{I} = \pm u_i. & - \leftrightarrow z<0 \end{array}\right. \tag{22}$$

3.1.2 Reduction to an Integral Equation

We will treat here only the determination of u_d^{I} which we will denote by u_d, without further definition. Let x be a point in space and let ξ be a point of the plane z = 0, and let us then pose /III,7

$$G(x, \xi) = \frac{1}{2\pi} \frac{e^{-ik|x-\xi|}}{|x-\xi|}, \tag{23}$$

considering the single-layer potential

$$\mathcal{T}^{I}(x) = \iint_{\sigma} f(\xi)\, G(x, \xi)\, dS_{\xi}, \tag{24}$$

Extended over the aperture, this represents a radiant solution of the Helmholtz equation outside of z = 0; this solution is continuous across the screen, which is also true of $\frac{\partial \mathcal{T}^{I}}{\partial t}$ which, incidentally, vanishes on the screen because of $\left(\frac{\partial G}{\partial t}\right)_{z=0} = 0$ for x differing from ξ.

Theorem 1: If the function $f(\xi)$, defined and continuous over the aperture, satisfies the integral equation

$$u_i(x_0) = \iint_{\sigma} f(\xi)\, G(x_0, \xi)\, dS_{\xi}, \qquad x_0 \in \sigma \tag{25}$$

and is such that $d^a f(\xi)$ remains bounded at $0 \le a < 1$, where d is the distance of ξ from the rim of the aperture, then the function

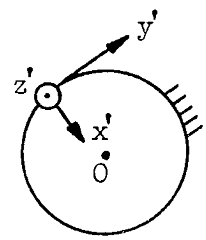
$$u_d(x) = \frac{z}{|z|} \iint_O f(\xi) \, \frac{e^{-iK|x-\xi|}}{|x-\xi|} \, dS_\xi \, . \qquad (26)$$

solves the first diffraction problem.

To understand the events taking place in the vicinity of the aperture rim it can be assumed (on condition that there is a technical justification /II... for this) that the boundary is locally rectilinear; we can then replace the Helmholtz potential by a Newtonian potential and replace f by its behavior. For the behavior of u_d in the vicinity of the boundary, this will lead to

$$u_d(x_0 + x') \cong \frac{A(x_0)}{2\pi} \int_0^1 \xi^{-a} \, d\xi \int_{-1}^{1} \frac{d\eta}{\{(x'-\xi)^2 + (y'-\eta)^2 + z'^2\}^{\frac{1}{2}}} \, , \qquad (27)$$

where x' , y' , z' are the coordinates of $x' + x_0$ in the system of axes of the accompanying diagram with $A(x_0) = \lim\limits_{d \to 0} d^a f(\xi)$ where the point ξ tends toward x_0



and where d is the distance to the tangent in x_0. This will yield

$$u_d \cong \frac{A(x_0)}{2\pi} \int_0^1 \xi^{-a} \, d\xi \left\{ \mathrm{Argsh} \left(\frac{\eta - y'}{\sqrt{(x'-\xi)^2 + z'^2}} \right) \right\}_{\eta=-1}^{\eta=+1} , \qquad (28)$$

and, consequently,

$$\nabla u_d \simeq \frac{A(x_0)}{2\pi} \int_0^1 d\xi \, \xi^{-a} \left\{ \frac{[(x'-\xi)^2 + z'^2]^{\frac{1}{2}} \, \nabla_{x'} \left(\frac{\eta - y'}{\sqrt{(x'-\xi)^2 + z'^2}} \right)}{\sqrt{(x'-\xi)^2 + (\eta - y')^2 + z'^2}} \right\}_{\eta=-1}^{\eta=+1} , \qquad (29)$$

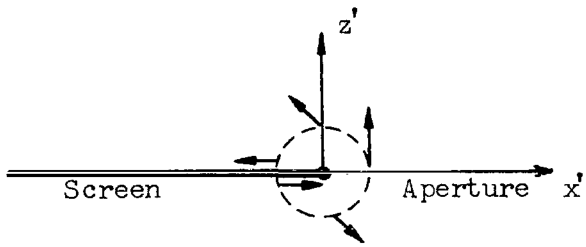
such that the most important component of the gradient coincides with the plane x' , z' ; if the calculation is limited to this component, we have

$$\nabla u_d \cong \frac{A(x_0)}{n} \bar{\nabla}_{x'}\left(\int_0^1 \xi^{-a} \operatorname{Log}\frac{1}{\sqrt{(x'-\xi)^2+z'^2}} d\xi\right) \tag{30}$$

as soon as x' and z' tend to zero, yielding

$$\begin{aligned}\frac{\partial u_d}{\partial x'} - i\frac{\partial u_d}{\partial z'} &= -\frac{A(x_0)}{n}\int_0^1 \frac{\xi^{-a}}{x'-\xi+iz'}\,d\xi \\ &\cong \frac{A(x_0)}{n}\left(x'+iz'\right)^{-a}.\end{aligned} \tag{31}$$

In the only known exact solution, it will be found that $a = \frac{1}{2}$ (here, a circular aperture is involved), so that it can be conceived that this result is general. In the accompanying diagram we have represented the evolution of the /III,9



component in a plane perpendicular to the boundary of the vector ∇u_d. On rotating about this boundary by describing a circle of radius ϵ, the modulus of ∇u_d will become equal to $\frac{|A(x_0)|}{n}\epsilon^{-\frac{1}{2}}$ and the diagram will correspond to $A(x_0) > 0$.

3.1.3 Energy Considerations

Let us pose $x = r\,\omega$ and let us study the behavior of u_d when $r \to \infty$; making use of

$$x - \xi \cong |x| - \xi\cdot\omega, \tag{32}$$

we find

$$u_d(r\omega) \cong \left\{\frac{z}{|z|}\frac{1}{2\pi}\iint_\sigma f(\xi)\, e^{ik(\omega\cdot\xi)}\, dS_\xi\right\}\frac{e^{-ikr}}{r}. \tag{33}$$

Posing

$$A(\omega) = \frac{z}{|z|}\,\frac{1}{2\pi} \iint_{\sigma} f(\xi)\, e^{ik\omega\cdot\xi}\, ds_{\xi}\,, \tag{34}$$

the density vector of the energy flux of the diffracted sound field, at a great distance, will become

$$W \cong \frac{1}{2}\,\rho k^2 c\, |A(\omega)|^2\, \omega\,, \tag{35}$$

such that the equal values of the energy flux emerging from the large /III,10
hemispheres located in z > 0 or z < 0 will be equal to

$$E_d = \frac{1}{2}\,\rho k^2 c \iint_{\omega_z > 0} |A(\omega)|^2\, d\omega\,. \tag{36}$$

Problem 3: By conveniently applying the energy identity, establish the formula

$$E_d = -\frac{\rho i k c}{4} \iint_{\sigma} \left(u_i^* \frac{\partial u_d}{\partial z} - u_i \frac{\partial u_d^*}{\partial z} \right)_{z = 0_+} dx\,dy \tag{37}$$

Problem 4: By applying the theorem 7 of Chapter II, demonstrate that

$$\left(\frac{\partial u_d}{\partial z} \right)_{z=+0} = f(x) \qquad\qquad x \in \sigma. \tag{38}$$

Problem 5: Since the incident sound field is a plane wave

$$u_i(x) = e^{-ik\,x\cdot\omega_i}\,, \tag{39}$$

establish the formula

$$E_d = \pi \rho k c\, \mathrm{Im}\{A(\omega_i)\} \tag{40}$$

Transfer factor τ:

Let

$$E_i = \frac{\rho i K c}{4} \iint_{O} \left(u_i^* \frac{\partial u_i}{\partial z} - u_i \frac{\partial u_i^*}{\partial z} \right)_{z=0} dS \tag{41}$$

be the energy flux which would traverse the aperture if the incident sound field were isolated, and let us define the transmission coefficient of the /III,11 aperture by the formula

$$\tau = \frac{E_d}{E_i \left(\iint_{\sigma} dS \right)^{-1}} . \tag{42}$$

Problem 6: For a plane incident wave of a direction ω_i, establish the relation

$$\tau = \frac{2\pi}{K} \, Im \; A(\omega_i) , \tag{43}$$

representing the Levine and Schwinger theorem.

3.1.4 Variational Principle of Levine and Schwinger

Below, let us consider an incident plane wave

$$u_i = e^{-iK x \cdot \omega_i} , \tag{44}$$

and let $f(\xi, \omega_i)$ be the unknown intensity of the single-layer potential which, by means of eq.(26), solves the diffraction problem whose solution is assumed to exist, i.e.,

$$e^{-iK \xi \cdot \omega_i} = \iint_{O} f(\xi', \omega_i) \; G(\xi, \xi') \; dS_{\xi'} , \tag{45}$$

where the points ξ, ξ' are both located on O. In accordance with the preceding §, we define $A(\omega_i, \omega)$ by

$$\left\{\begin{array}{l} A(\omega_i,\omega) = \frac{1}{2\pi}\iint\limits_{\sigma} f(\xi,\omega_i)\, e^{iK\omega\cdot\xi}\, dS_{\xi}\,, \qquad \omega_z > 0 \\ A(\omega_i,-\omega) + A(\omega_i,\omega) = 0\,. \end{array}\right. \tag{46}$$

Let us then consider a second incident plane wave of a direction $-\omega_i'$, where $f(\xi, -\omega_i')$ is the corresponding density, thus yielding

/III,12

$$\iint\limits_{\sigma}\iint\limits_{\sigma} dS_{\xi}\, dS_{\xi'}\, f(\xi,-\omega_i')\, G(\xi,\xi')\, f(\xi',\omega_i) = \iint\limits_{\sigma} f(\xi,-\omega_i')\, e^{-iK\xi\cdot\omega_i}\, dS_{\xi} \tag{47}$$

from which we obtain the formula

$$A(\omega_i,\omega_i') = A(-\omega_i',-\omega_i) = \tag{48}$$
$$= \frac{1}{2\pi}\,\frac{\iint\limits_{\sigma} f(\xi',\omega_i)\, e^{iK\omega_i'\cdot\xi'}\, dS_{\xi'} \iint\limits_{\sigma} f(\xi,-\omega_i')\, e^{-iK\omega_i\cdot\xi}\, dS_{\xi}}{\iint\limits_{\sigma}\iint\limits_{\sigma} dS_{\xi}\, dS_{\xi'}\, f(\xi,-\omega_i')\, G(\xi,\xi')\, f(\xi',\omega_i)}\,.$$

Consequently, for the transfer factor, we obtain the following expression:

$$\tau(\omega) = \tau(-\omega) = \tag{49}$$
$$\frac{1}{4K}\,\mathrm{Im}\,\frac{\iint\limits_{\sigma} f(\xi',\omega)\, e^{iK\omega\cdot\xi'}\, dS_{\xi'} \iint\limits_{\sigma} f(\xi,-\omega)\, e^{-iK\omega\cdot\xi}\, dS_{\xi}}{\iint\limits_{\sigma}\iint\limits_{\sigma} dS_{\xi}\, dS_{\xi'}\, f(\xi,-\omega)\, G(\xi,\xi')\, f(\xi',\omega)}\,.$$

Theorem 2 (Levine and Schwinger): The second members of eqs.(48) and (49) are stationary relative to any infinitesimal modification of f about the solution of the integral equation (45).

Let us write eq.(48) in the form of

$$0 = I \equiv \iint_{\sigma}\iint_{\sigma} dS_{\xi}\, dS_{\xi'}\, f(\xi', \omega_i)\left\{ e^{iK[\omega_i' \cdot \xi' - \omega_i \cdot \xi]} - 2\pi A(\omega_i, \omega_i')\, G(\xi, \xi') \right\} f(\xi, \omega_i') \tag{50}$$

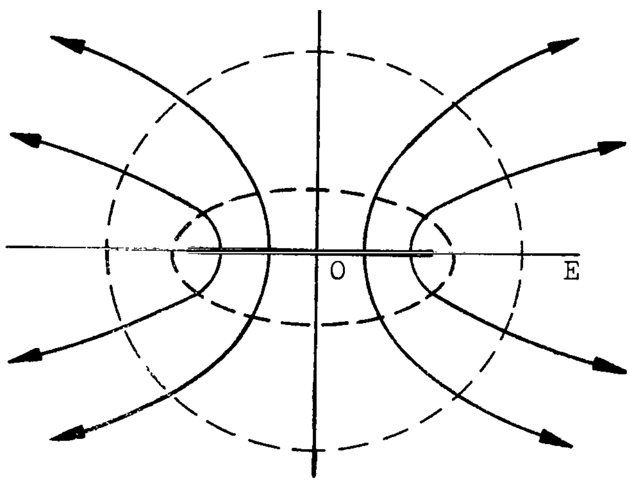
and let us give, to f, a variation δf by leaving A fixed; we find that $\delta I = 0$ in first order, as a consequence of eq.(45) and of the analogous relation CQFD.

3.1.5 Diffraction by a Small Aperture /III,13

Let 2a be the maximum diameter of the aperture if $ka \ll 1$; we can then attempt to bring eq.(25) to the form

$$\frac{1}{2\pi}\iint_{\sigma} \frac{f(\xi)}{|x-\xi|}\, dS_{\xi} = u_i(x) + \frac{1}{2\pi}\iint_{\sigma} f(\xi)\, \frac{1 - e^{-iK|x-\xi|}}{|x-\xi|}\, dS_{\xi}, \tag{51}$$

with the purpose of treating the integral of the second term as a small perturbation.



Thus, let us consider the integral equation

$$\frac{1}{2\pi}\iint_{\sigma} \frac{g(\xi)}{|x-\xi|}\, dS_{\xi} = h(x), \tag{52}$$

and then let us attempt to interpret its solution, which is supposed to exist. For x outside of O,

$$V(x) = \frac{1}{2\pi}\iint_{\sigma} \frac{g(\xi)}{|x-\xi|}\, dS_{\xi}\,, \tag{53}$$

is a solution of the Laplace equation $\Delta V = 0$, which has a value of $h(x)$ on O and is such that $\frac{dV}{dn}$ assumes a value of $-g(x)$ on O and vanishes on E. If $h(x) = V_c$ = const, then $V(x)$ represents the field of electric potential created by a metal conductor, infinitely spread over the aperture and brought to the potential V_c, on which a surface charge density $\sigma = -\frac{1}{4\pi}\frac{dV}{dn}$ develops. If $u_i(x)$ is a plane wave parallel to O, this interpretation will hold.

Let us attempt to interpret the hypothesis that $ka \ll 1$. Let us place the origin of the coordinates at the center of the aperture and let us pose

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$$\begin{cases} x = a\,\underset{\sim}{x}\,, \quad \xi = a\,\underset{\sim}{\xi}\,, \quad dS_{\xi} = a^2\, d\underset{\sim}{S}_{\xi}\,, \\ f(a\underset{\sim}{\xi}) = a^{-1}\,\underset{\sim}{f}(\underset{\sim}{\xi})\,, \\ \dfrac{1-e^{-ikr}}{ikr} = \mathcal{E}(ikr)\,, \end{cases} \tag{53}$$

so that

$$\frac{1}{2\pi}\iint_{\underset{\sim}{\sigma}} \frac{\underset{\sim}{f}(\underset{\sim}{\xi})}{|\underset{\sim}{x}-\underset{\sim}{\xi}|}\, d\underset{\sim}{S}_{\xi} = u_i(a\underset{\sim}{x}) + \frac{ika}{2\pi}\iint_{\underset{\sim}{\sigma}} \underset{\sim}{f}(\underset{\sim}{\xi})\, \mathcal{E}(ika|\underset{\sim}{x}-\underset{\sim}{\xi}|)\, d\underset{\sim}{S}_{\xi}\,, \tag{54}$$

which shows that, if ka is very small, it can actually be attempted to construct $\underset{\sim}{f}(\underset{\sim}{\xi})$ by an iterative process based on the above analogy. To be more accurate, let us assume that $u_i(x)$ is a plane wave so that $u_i(\underset{\sim}{x})$ can be expanded in powers of ika:

$$u_i(x) = u_i(0)\sum_{h=0}^{\infty} (-ika)^n \frac{(r\cos\theta)^n}{n!} \tag{55}$$

with notations still to be defined; in the same way, $\mathcal{E}(ika|\underset{\sim}{x}-\underset{\sim}{\xi}|)$ can be expanded in powers of ika:

$$\mathcal{E}\left(ika\,|\underset{\sim}{x}-\underset{\sim}{\xi}|\right) = 1 - \frac{ika}{2}\,|\underset{\sim}{x}-\underset{\sim}{\xi}| + \sum_{n=2}^{\infty} (-ika)^{n}\,\frac{|x-\xi|^{n}}{(n+1)!}\,, \tag{56}$$

and it can be conceived that $f(\xi)$ itself can be expanded in powers of ika. We will not attempt here to study the convergence of such a series and do not even pretend to construct it effectively; we will be satisfied to center our attention on the very first terms by assuming $u_1 = 1$, i.e., we will limit the calculation to the case of a plane wave, incident parallel to the aperture.

Let

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$$\underset{\sim}{f}(\underset{\sim}{\xi}) = \sum_{n=0}^{N} (ika)^{n}\,\underset{\sim}{f}_{n}(\underset{\sim}{\xi}) + R_N \tag{57}$$

be the limited expansion in question, yielding

$$\underset{\sim}{f}_0 = 4\pi\,\underset{\sim}{\sigma}(\underset{\sim}{\xi}), \tag{58}$$

where $\underset{\sim}{\sigma}(\underset{\sim}{\xi})$ denotes the surface charge density of a metal disk, coinciding with the aperture and brought to unit potential, with the zero potential being that of infinity. Then, the integral of the second term has a value of ika $\underset{\sim}{Q}$, where $\underset{\sim}{Q}$ denotes the charge of the (two-face) conductor, where a is selected as unit length in evaluating $\underset{\sim}{Q}$. Under these conditions, we have

$$\underset{\sim}{f}_0 + ika\,\underset{\sim}{f}_1 = 4\pi\left(1 + ika\,\underset{\sim}{Q}\right)\underset{\sim}{\sigma}(\underset{\sim}{\xi}), \tag{59}$$

which we will retain below since, for the subsequent terms, the electric analogy loses its excellent simplicity!

Let us substitute eq.(59) into eq.(49), obtaining the following expression for the transmission factor:

$$\tau = \frac{a}{4K}\,\mathrm{Im}\,\frac{\left(\iint_O \underset{\sim}{\sigma}(\underset{\sim}{\xi})\,dS_{\xi}\right)^{2}}{\frac{1}{2\pi}\iint_O\iint_O \underset{\sim}{\sigma}(\underset{\sim}{\xi})\,\dfrac{e^{-ika|\xi-\xi'|}}{|\underset{\sim}{\xi}-\underset{\sim}{\xi}'|}\,\underset{\sim}{\sigma}(\underset{\sim}{\xi}')\,dS_{\xi}\,dS_{\xi'}} \tag{60}$$

The theorem 2 guarantees that, if this expression is expanded in powers of ika and if the series is terminated at $(ka)^2$, we will have obtained a limited expan-

sion of the exact value of τ up to the term in $(ka)^2$ inclusive. In accordance with the expansion of the imaginary exponential in the denominator, we find /III,16

$$\tau = \frac{n a^2}{2} \frac{\left(\iint_{O} \sigma \, dS\right)^4}{\left(\iint_{O}\iint_{O} \sigma \frac{1}{r} \sigma' dS \, dS'\right)^2 + k^2 a^2 \left\{\left(\iint_{O} \sigma \, dS\right)^4 - \iint_{O}\iint_{O} \sigma \frac{1}{r} \sigma' dS \, dS' \iint_{O}\iint_{O} \sigma \, r \, \sigma' dS \, dS'\right\}} \tag{61}$$

denoting by r the distance $|\xi - \xi'|$.

Since we have set $u_1 = 1$, this means that the potential of the metal conductor has been taken as equal to unity, so that $\iint_{O} \sigma \, dS = \frac{1}{2} Q = \frac{1}{2} \Gamma$ where $a\Gamma$ denotes the capacitance of the (two-face) conductor. In an analogous manner, we find

$$\iint_{O}\iint_{O} \sigma \frac{1}{r} \sigma' dS \, dS' = \frac{1}{2} \iint_{O} \sigma \, dS = \frac{1}{4} \Gamma, \tag{62}$$

such that, in first approximation,

$$\tau_0 = \frac{n a^2}{2} \Gamma^2 \simeq \frac{n}{2} \Gamma^2 . \tag{63}$$

The transfer factor of a small plane aperture ($ka \ll 1$), in first approximation, is $\frac{n}{2}$ times the capacitance of a metal conductor spread over the aperture. If the expansion is continued for one more term, as we are entitled to do, the following result will be obtained:

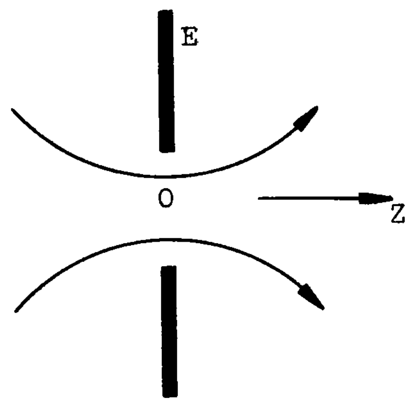
$$\tau = \frac{n \Gamma^2}{2} \left\{ 1 + k^2 a^2 \left[\frac{\iint_{O}\iint_{O} \sigma \, r \, \sigma' dS \, dS'}{\iint_{O}\iint_{O} \sigma \frac{1}{r} \sigma' dS \, dS'} - \Gamma^2 \right] + O(k^4 a^4) \right\} \tag{64}$$

so that the capacitance no longer is the only parameter here.

Problem 7: Let us consider a steady-state flow of an incompressible ideal fluid across the aperture O, tending toward rest on either side of infinity. Let ϕ be the velocity potential of this irrotational motion, posing /III,17

$$[\phi] = +\phi_{\infty \text{ upstream}} - \phi_{\infty \text{ downstream}} , \tag{65}$$

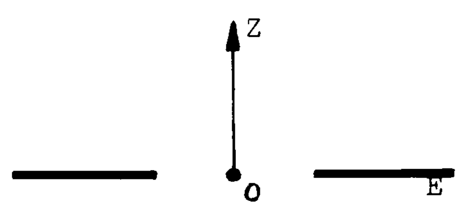
denoting by $\rho_0 \underset{\sim}{Q}$ the mass flux across the aperture. Demonstrate that the kinetic



energy of motion is given by

$$\mathcal{T} = \frac{1}{2} \rho_0 Q [\phi] . \tag{66}$$

Demonstrate that $[\phi]$ is still equal to the discontinuity of ϕ across the screen E, at any point of the latter.



<u>Problem 8</u>: Let ϕ_1 be defined in $z > 0$, such that

$$\left\{ \begin{array}{lr} \Delta \phi_1 = 0 , \qquad z > 0 & \\ \phi_1 = \frac{1}{2} [\phi] = \text{Const} , & \text{on E} \\ \dfrac{d\phi_1}{dz} = 0 , & \text{on E} \\ \phi_1 = \frac{1}{2} [\phi] , & \text{at infinity} \end{array} \right. \tag{67}$$

and let, similarly, ϕ_2 be defined by

$$\begin{cases} \Delta \phi_2 = 0 , & z > 0 \\ \phi_2 = \frac{1}{2}[\phi] = \text{const} , & \text{at } E , \\ \frac{d\phi_2}{dz} = 0 , & \text{at } E , \\ \phi_2 = 0 , & \text{at infinity} . \end{cases} \tag{68}$$

Demonstrate that $\phi_1 + \phi_2 = \frac{1}{2}[\phi]$ = const and derive from this

$$\iint_\sigma \frac{d\phi_1}{dn} dS = \pi \Gamma [\phi] \tag{69}$$

where Γ is the capacitance of the conductor spread over O.

<u>Problem 9</u>: Using the notations of problem 5, demonstrate that /III,18

$$\tau = \frac{\pi}{2} \Gamma \rho_0 [\phi]^2 = \frac{\pi}{2\Gamma} \rho_0 Q^2 . \tag{70}$$

<u>Problem 10</u>: Demonstrate that, if $ka \ll 1$ at a point located a finite distance from the aperture, we will have

$$u_d \cong \frac{Q}{2\pi} \frac{e^{-ikr}}{r} . \tag{71}$$

Demonstrate that $\rho_0 Q$ can be interpreted as the outflow of mass across the aperture. Demonstrate that the energy flux E_d, radiated toward the right through the aperture, is correlated with the kinetic energy of the incompressible flow with an outflow of Q across the aperture, by the relation

$$E_d = \frac{\Gamma k c^2}{16 \pi^2} \tau . \tag{72}$$

3.1.6 Diffraction by a Small Elliptic Aperture

The aperture is defined by

$$\frac{x^2}{\alpha^2} + \frac{y^2}{\beta^2} - 1 \leq 0 , \qquad z = 0 , \tag{73}$$

in reduced coordinates (with x replacing $\underset{\sim}{x}$). This forms part of the family of homofocal quadrics

$$\frac{x^2}{\alpha^2+\rho}+\frac{y^2}{\beta^2+\rho}+\frac{z^2}{\rho}=1, \tag{74}$$

and corresponds to $\rho = 0_+$ while E corresponds to $\rho = 0_-$. At a point (x, y, z), three quadrics of the family are passed, corresponding to the values λ_1, λ_2, λ_3 of ρ, yielding

$$1-\left(\frac{x^2}{\alpha^2+\rho}+\frac{y^2}{\beta^2+\rho}+\frac{z^2}{\rho}\right)=\frac{(\rho-\lambda_1)(\rho-\lambda_2)(\rho-\lambda_3)}{(\alpha^2+\rho)(\beta^2+\rho)\rho}\,. \tag{75}$$

Problem 11: Establish the formulas /III,19

$$\left\{\begin{aligned} x^2&=\frac{(\lambda_1+\alpha^2)(\lambda_2+\alpha^2)(\lambda_3+\alpha^2)}{\alpha^2(\alpha^2-\beta^2)},\\ y^2&=\frac{(\lambda_1+\beta^2)(\lambda_2+\beta^2)(\lambda_3+\beta^2)}{\beta^2(\beta^2-\alpha^2)},\\ z^2&=\frac{\lambda_1\lambda_2\lambda_3}{\alpha^2\beta^2}, \end{aligned}\right. \tag{76}$$

$$\begin{aligned} 4\left(dx^2+dy^2+dz^2\right)&=\left\{\frac{x^2}{(\lambda_1+\alpha^2)^2}+\frac{y^2}{(\lambda_1+\beta^2)^2}+\frac{z^2}{\lambda_1^2}\right\}d\lambda_1^2+\left\{\frac{x^2}{(\lambda_2+\alpha^2)^2}+\frac{y^2}{(\lambda_2+\beta^2)^2}+\frac{z^2}{\lambda_2^2}\right\}d\lambda_2^2\\ &+\left\{\frac{x^2}{(\lambda_3+\alpha^2)^2}+\frac{y^2}{(\lambda_3+\beta^2)^2}+\frac{z^2}{\lambda_3^2}\right\}d\lambda_3^2=\frac{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}{(\alpha^2+\lambda_1)(\beta^2+\lambda_1)\lambda_1}\,d\lambda_1^2+\\ &+\frac{(\lambda_2-\lambda_1)(\lambda_2-\lambda_3)}{(\alpha^2+\lambda_2)(\beta^2+\lambda_2)\lambda_2}\,d\lambda_2^2+\frac{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}{(\alpha^2+\lambda_3)(\beta^2+\lambda_3)\lambda_3}\,d\lambda_3^2\,. \end{aligned} \tag{77}$$

Let us consider the function

$$\phi=\int_0^{\lambda_1}\frac{d\lambda}{\sqrt{(\alpha^2+\lambda)(\beta^2+\lambda)\lambda}}, \tag{78}$$

whose gradient obviously is directed along the tangent to the line λ_2 = const, λ_3 = const, having the following modulus value:

$$|\nabla\phi| = 2\left\{(\alpha^2+\lambda_1)(\beta^2+\lambda_1)\lambda_1\left(\frac{x^2}{(\alpha^2+\lambda_1)^2}+\frac{y^2}{(\beta^2+\lambda_1)^2}+\frac{t^2}{\lambda_1^2}\right)\right\}^{-1/2} = \frac{2}{\sqrt{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}} \tag{79}$$

in such a manner that the flux of $\nabla\phi$ through the cross section of a tube formed by the lines λ_2 = const, λ_3 = const will be maintained when λ_1 varies. From this, it follows that the function ϕ is harmonic. It is obvious that $\frac{d\phi}{dn}$ vanishes on E and that ϕ vanishes on O, whereas, at infinity, ϕ assumes a constant value of

$$I(\alpha,\beta) = \int_0^\infty \frac{d\lambda}{\sqrt{(\alpha^2+\lambda)(\beta^2+\lambda)}}, \tag{80}$$

such that $\pm\,\phi$ solves the problem 7, with $\frac{1}{2}[\phi]$ = I. Now, let us note /III,20
that for very large λ_1, the ellipsoid $\rho = \lambda_1$ is reduced to a sphere of radius $\sqrt{\lambda_1}$ in such a manner that, at infinity, we will approximately have

$$\phi = -\frac{2}{r} + I, \tag{81}$$

and that the outflow of mass through a large hemisphere will be

$$\rho_0 Q = 4\pi\rho_0 = 2\pi\rho\underset{\sim}{\Gamma} I, \tag{82}$$

which yields the value of $\underset{\sim}{\Gamma}$ for an elliptic aperture:

$$\underset{\sim}{\Gamma}^{-1} = \frac{1}{2}\int_0^\infty \frac{d\lambda}{\sqrt{(\alpha^2+\lambda)(\beta^2+\lambda)\lambda}} = \int_0^{\pi/2}\frac{d\theta}{\sqrt{\alpha^2\sin^2\theta+\beta^2\cos^2\theta}}. \tag{83}$$

<u>Problem 12</u>: Demonstrate that, for an elliptic aperture with axes 2a and 2b, the transfer factor will be

$$\tau_0 = \frac{\pi a^2\alpha^2}{2F^2(e)}\frac{4}{\pi^2} \qquad e^2 = \frac{\alpha^2-\beta^2}{\alpha^2} \tag{84}$$

where F(e) denotes the elliptic integral

$$F(e) = \frac{2}{\pi}\int_0^{\pi/2}\frac{d\theta}{\sqrt{1-e^2\cos^2\theta}}. \tag{85}$$

e	0	0.5	0.766	0.866	0.940	0.955	1
F(e)	1	1.0013	1.0122	1.0301	1.0724	1.1954	∞

(86)

Because of eq.(79), it is obvious that, on 0, we have

$$\frac{d\phi}{dn} = \frac{2}{\alpha\beta\sqrt{1-\frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2}}}, \tag{87}$$

such that the surface charge density $\underset{\sim}{\sigma}$, introduced in the preceding Section, will be

$$\underset{\sim}{\sigma} = \frac{\underset{\sim}{\Gamma}}{4\pi\alpha\beta}\,\frac{1}{\sqrt{1-\frac{x^2}{\alpha^2}-\frac{y^2}{\beta^2}}}, \tag{88}$$

after which it can be proved that /III,21

$$\iint_{\underset{\sim}{\sigma}} \underset{\sim}{\sigma}\, d\underset{\sim}{x}\, d\underset{\sim}{y} = \frac{1}{2}\underset{\sim}{\Gamma}. \tag{89}$$

The case of a circular aperture, of dimensions that are not necessarily small, has been calculated exactly by Bouwkamp, using the method of separation of variables; he gave the following expression:

$$\begin{aligned}\tau = {} & \frac{8}{\pi^2}\Big\{1+\Big(\frac{4}{9}-\frac{4}{\pi^2}\Big)(ka)^2+\Big(\frac{71}{675}-\frac{8}{3\pi^2}+\frac{16}{\pi^4}\Big)(ka)^4+ \\ & +\Big(\frac{568}{33075}-\frac{1536}{2025\pi^2}+\frac{128}{9\pi^4}-\frac{64}{\pi^6}\Big)(ka)^6+\dots\Big\} \\ = {} & 0.810569\{1+0.039160(ka)^2-0.0007489(ka)^4-0.0002602(ka)^6+\dots\}\end{aligned} \tag{90}$$

The same author also calculated the function A(ω) which, in our case (plane wave incident parallel to the plate), depends only on the angle θ included by the vector ω and the normal to the plate, namely,

$$\frac{2\pi}{a}A(\theta) = 1+\Big(\frac{1}{3}-\frac{4}{\pi^2}-\frac{1}{6}\sin^2\theta\Big)(ka)^2+\Big\{\frac{16}{\pi^4}-\frac{20}{9\pi^2}+\frac{7}{15}+ \tag{91}$$

$$+\left(\frac{2}{3\pi^2}-\frac{1}{15}\right)\sin^2\theta+\frac{1}{120}\sin^4\theta\Big\}(ka)^4-\frac{2ika}{\pi}\left\{1+\left(\frac{4}{3}-\frac{4}{\pi^2}-\frac{1}{6}\sin^2\theta\right)(ka)^2\right\}$$
$$+\,o\left((ka)^5\right).$$

3.1.7 Numerical Results

Bouwkamp made a numerical calculation, on the basis of his explicit formulas, of the exact value of τ for a circular aperture of radius a, at a plane wave incident plane-parallel to the plate; he gave the following results:

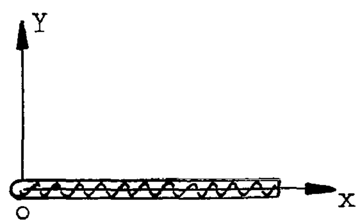
ka	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	
τ	0.8106	0.8109	0.8118	0.8134	0.8156	0.8185	0.8219	0.8260	0.8306	0.8358	
ka	1.0	1.2	1.4	1.6	1.8	2.0	2.2	2.4	2.6	2.8	
τ	0.8415	0.854	0.868	0.883	0.900	0.917	0.932	0.945	0.958	0.968	
ka	3.0	3.2	3.4	3.6	3.8	4.0	4.2	4.4	4.6	4.8	5.0
τ	0.976	0.982	0.966	0.988	0.988	0.987	0.987	0.986	0.986	0.987	0.988
								6.00	8.00	10.0	
								0.993	0.996	1.0	

When ka $\gg$ 1, geometric acoustics (which will be discussed in Chapter IV) indicates that the sound field passes through the aperture like a light beam such that τ = 1, as can also be seen in the Table given in the preceding Section. In the geometric wake or shadow of the aperture, there is absolute silence. In fact, in acoustics, the case ka $\gg$ 1 is very rarely realized! /III,22

3.2 Diffraction by a Half-Plane

3.2.1 Multivalent Helmholtz Functions

We will consider here only functions of two variables x and y and will construct functions u(x, y) that are solutions of $\Delta u + k^2 u = 0$, taking different



values on either side of the positive semiaxis ox. These functions will be used later for solving the Sommerfeld problem, consisting in finding a Helmholtz function whose derivative $\frac{\partial u}{\partial y}$ vanishes in $y = \pm 0$, $x > 0$, and has a suitable behavior at infinity.

We will make use of polar coordinates

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta; \end{cases} \tag{1}$$

and note that

$$e^{ikr\cos(\theta-\alpha)} \tag{2}$$

is a solution of the Helmholtz equation. This is also true of

$$\int_L f(\alpha)\, e^{ikr\cos(\theta-\alpha)}\, d\alpha, \tag{3}$$

where L is a suitable path. It is sufficient that L does not depend on /III,23
ρ or θ and that differentiation under the integral sign is permitted. The path L can be traced in the complex plane, a procedure which might even be quite advantageous.

Thus, let $\zeta = \xi + i\eta$ and let us take, for L in the plane of the variable ζ, a small closed contour surrounding the point $\theta_0 - \theta$; the Cauchy theorem then guarantees that we have

$$\begin{aligned} e^{ikr\cos(\theta-\theta_0)} &= \frac{1}{2\pi}\oint_L e^{ikr\cos\zeta}\,\frac{e^{i\zeta}}{e^{i\zeta}-e^{i(\theta_0-\theta)}}\,d\zeta \\ &= \frac{1}{2\pi}\oint_L e^{ikr\cos(\theta-\alpha)}\,\frac{e^{i\alpha}}{e^{i\alpha}-e^{i\theta_0}}\,d\alpha. \end{aligned} \tag{4}$$

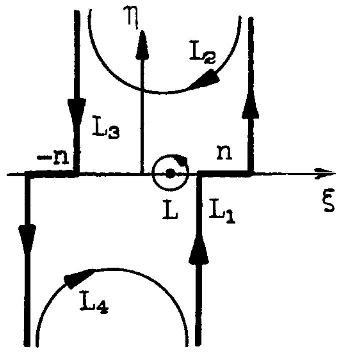
The contour L may be deformed into the contour $L_1 + L_2 + L_3 + L_4$ of the accompanying diagram, without changing the value of the integral, under the provision that $\mathrm{Re}(i\cos\zeta) \to -\infty$ along the infinite branches, which takes place if, along these branches, we have

$$\zeta = (\pm n + \varepsilon \operatorname{sgn} \eta) + i\eta , \tag{5}$$

where $\varepsilon(\eta)$ tends toward a nonzero finite limit, inferior to n, as $|\eta| \to \infty$, in view of the fact that

$$Re \left(i \cos\{(\pm n + \varepsilon \operatorname{sgn} \eta) + i\eta\} \right) = - \sin(\varepsilon \operatorname{sgn} \eta) \operatorname{sh} \eta . \tag{6}$$

It is obvious that $\int_{L_1} + \int_{L_3} = 0$, in such a manner that, if A denotes the combination of the paths L_2 and L_3 traversed in a direction opposite to /III,24



that of the arrows, we will obtain

$$e^{ik r \cos(\theta - \alpha)} = \frac{1}{2n} \int_A e^{ikr \cos\zeta} \frac{e^{i\zeta}}{e^{i\zeta} - e^{i(\theta_0 - \theta)}} d\zeta . \tag{7}$$

Let us now consider, based on the structure of the preceding formula, the following function:

$$\begin{aligned} u &= \frac{1}{4n} \int_A e^{ikr \cos\zeta} \frac{e^{i\zeta/2}}{e^{i\zeta/2} - e^{i(\theta_0 - \theta)/2}} d\zeta \\ &= \frac{1}{8n} \int e^{ikr \cos(\theta - 2\alpha)} \frac{e^{i\alpha}}{e^{i\alpha} - e^{i\frac{\theta_0}{2}}} d\alpha , \end{aligned} \tag{8}$$

It is obvious, under the second form, that u is a Helmholtz function since it is evident, under the first form, that this could be derived under the integral sign.

The following relation

$$u(r,\theta) + u(r,\theta+2\pi) = e^{ikr\cos(\theta-\theta_0)} \tag{9}$$

becomes of great importance if it is combined with the following:

$$u(r,\theta) \to 0 \quad , \quad r \to \infty \quad , \quad \theta_0 + 2\pi < \theta < \theta_0 + 4\pi \quad , \tag{10}$$

since it establishes that u does not admit the period 2n in θ but that it does admit the period 4n in θ and that, consequently, u will assume two different values at each point of the plane, excluding the origin. Below and until we formally state the theorem 3, we will use $\theta_0 < \theta < \theta_0 + 2n$. To establish eq.(9), let us consider the contour A' as being derived from A by the translation -2n; maintaining the direction of the course, we will have

$$u(r,\theta+2\pi) = \frac{1}{4\pi}\int_{A'} e^{ikr\cos\zeta}\,\frac{e^{i\zeta/2}}{e^{i\zeta/2} - e^{i(\theta_0-\theta)/2}}\,d\zeta, \tag{11}$$

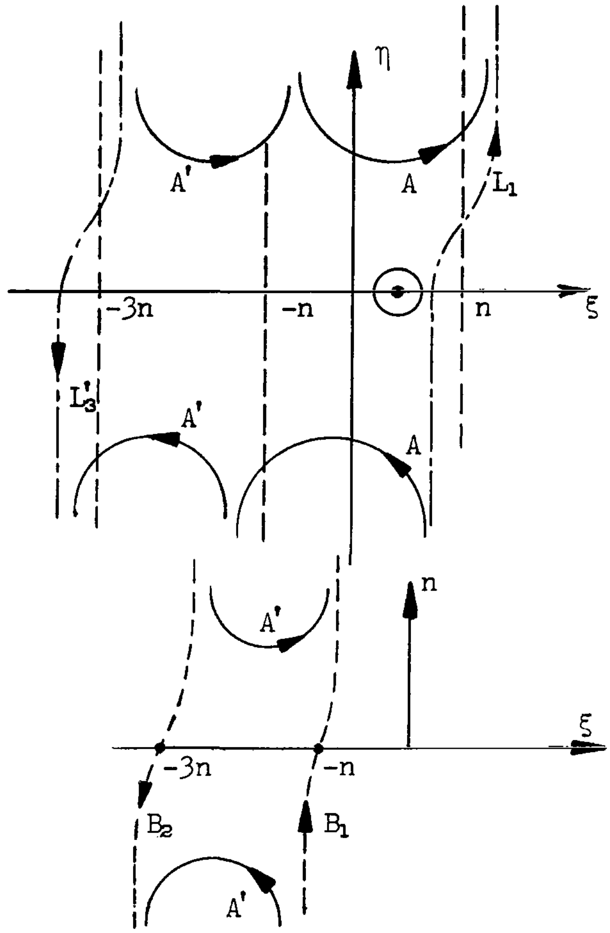
so that eq.(9) results from /III,25

$$e^{ikr\cos(\theta-\theta_0)} = \frac{1}{4\pi}\int_{A+A'+L_1+L'_3} e^{ikr\cos\zeta}\,\frac{e^{i\zeta/2}}{e^{i\zeta/2} - e^{i(\theta_0-\theta)/2}}\,d\zeta, \tag{12}$$

as demonstrated by Cauchy's theorem since, as proved, $\int_{L_1} + \int_{L'_3} = 0$, whereas the single pole at the interior of the total contour is $\zeta = \theta_0 - \theta$, with a residue which precisely yields eq.(12). Let us then complete A' by the path B covered in a direction inverse to that shown in the accompanying diagram; Cauchy's theorem then demonstrates that

$$u(r,\theta+2\pi) = \frac{1}{4\pi}\int_{B} e^{ikr\cos\zeta}\,\frac{e^{i\zeta/2}}{e^{i\zeta/2} - e^{i(\theta_0-\theta)/2}}\,d\zeta. \tag{13}$$

It is always possible to select B (see the diagram) in such a manner that,



along this path,

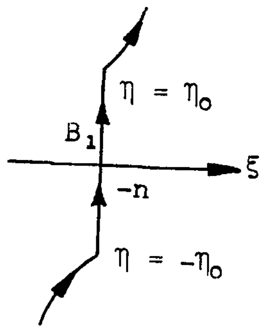
$$\mathrm{Re}\,(i\kappa\cos\zeta) \leq 0\,, \tag{14}$$

where equality takes place only in isolated points. This remark renders the relation (10) evident. Let us decompose B into $B_1 + B_2$ and let us transpose B_2 in B_1, yielding

$$u = \frac{1}{2\pi}\int_{B_1} e^{i\kappa r\cos\zeta}\,\frac{e^{i\zeta/2}\,e^{i(\theta_0-\theta)/2}}{e^{i\zeta} - e^{i(\theta_0-\theta)}}\,d\zeta\,. \tag{15}$$

Selecting B_1 as indicated in the accompanying diagram and making η_0 tend toward infinite, will yield

$$u = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikr\coth\eta} \, \frac{e^{-\eta/2} e^{i(\theta_0-\theta)/2}}{e^{-\eta} - e^{i(\theta_0-\theta)}} \, d\eta \tag{16}$$



and, finally,

/III,26

$$u = \frac{1}{\pi} \, e^{ikr\cos(\theta-\theta_0)} \int_0^{\infty} e^{-ikr[\coth\eta + \cos(\theta-\theta_0)]} \, \frac{\coth(\eta/2)\cos\left(\frac{\theta_0-\theta}{2}\right)}{\coth\eta + \cos(\theta_0-\theta)} \, d\eta. \tag{17}$$

Let us pose

$$\varpi(\rho, \theta; \theta_0) = e^{ikr\cos(\theta-\theta_0)}, \tag{18}$$

$$X = \frac{1}{\pi} \int_0^{\infty} e^{-ikr[\coth\eta + \cos(\theta-\theta_0)]} \, \frac{\coth(\eta/2)\cos[(\theta_0-\theta)/2]}{\coth\eta + \cos(\theta_0-\theta)} \, d\eta, \tag{19}$$

and let us attempt to transform X. For this, setting $\tau = \sinh(\eta/2)$, we will obtain

$$X = \frac{2}{\pi} \cos\left[\frac{\theta-\theta_0}{2}\right] \int_0^{\infty} e^{-ikr[2\tau^2+1+\cos(\theta-\theta_0)]} \, \frac{d\tau}{2\tau^2+1+\cos(\theta-\theta_0)}, \tag{20}$$

and, consequently,

$$\frac{\partial X}{\partial r} = -\frac{2ik}{\pi}\, e^{-ikr[1+\cos(\theta-\theta_0)]} \cos\frac{\theta-\theta_0}{2} \int_0^{\infty} e^{-2ikr\tau^2}\, d\tau$$

$$= e^{-i\pi/4} \sqrt{\frac{k}{2\pi r}}\, \cos\left[\tfrac{1}{2}(\theta-\theta_0)\right] e^{-2ikr\cos[(\theta-\theta_0)/2]}, \tag{21}$$

because of

$$\int_0^{\infty} e^{-i\lambda^2}\, d\lambda = \frac{\sqrt{\pi}}{2}\, e^{-\frac{\pi i}{4}}. \tag{22}$$

<u>Theorem 3</u>: Let us pose

$$\mathcal{F}(\lambda) = \frac{e^{\pi i/4}}{\sqrt{\pi}} \int_{-\infty}^{\lambda} e^{-iu^2}\, du . \tag{23}$$

The function

$$\left\{ \begin{aligned} &\Sigma(r,\theta;\theta_0) = \varpi(r,\theta;\theta_0)\, \mathcal{F}\left\{\sqrt{2kr}\, \cos\left(\frac{\theta-\theta_0}{2}\right)\right\}, \qquad \theta_0 \le \theta \le \theta_0 + 4\pi \\ &\varpi(r,\theta;\theta_0) = e^{ikr\cos(\theta-\theta_0)}, \end{aligned} \right. \tag{24}$$

is a solution of the Helmholtz equation which admits the period 4n in θ. This solution proves

/III,27

$$\left\{ \begin{aligned} &\Sigma(r,\theta;\theta_0) + \Sigma(r,\theta+2\pi;\theta_0) = \varpi(r,\theta;\theta_0), \\ &\lim_{r\to\infty} \left\{\Sigma(r,\theta;\theta_0) - \varpi(r,\theta;\theta_0)\right\} = 0, \qquad 0 < \theta-\theta_0 < 2\pi \\ &\lim_{r\to\infty} \Sigma(r,\theta;\theta_0) = 0 . \qquad 2\pi < \theta-\theta_0 < 4\pi \end{aligned} \right. \tag{25}$$

It is noted that $\frac{\partial \mathfrak{I}}{\partial r} = \frac{\partial X}{\partial r}$ and that

$$\mathcal{F}(\lambda) + \mathcal{F}(-\lambda) = \mathcal{F}(\infty) = 1 \tag{26}$$

in such a manner that the theorem can be proved if it can be demonstrated that

$\mathfrak{F} = X$ for a value of r and a value of θ such that $2n < \theta - \theta_0 < 4n$. It is sufficient to set $\cos \frac{\theta - \theta_0}{2n} < 0$ and $r = \infty$. It is not necessary to restrict θ in eq.(24), but it is convenient to do so, as indicated above.

3.2.2 Solution of the Sommerfeld Problem

Let us next return to the problem posed at the beginning of Section 2.1 and let us attempt to restrict the solution by a suitable combination of functions Σ. Let us first calculate the derivative $\frac{\partial \Sigma}{\partial y}$ on the screen

$$\left(\frac{\partial \Sigma(r,\theta;\theta_0)}{\partial y}\right)_{\substack{y=0\\x>0}} = \left\{ ik\, F\left(\sqrt{2kr}\cos\frac{\theta_0}{2}\right)\sin\theta_0 + \sqrt{\frac{k}{2r}}\, F'\left(\sqrt{2kr}\cos\frac{\theta_0}{2}\right)\sin\frac{\theta_0}{2}\right\} e^{ikr\cos\theta_0} \tag{27}$$

and let us attempt to form a function combination Σ for which the corresponding derivative vanishes; this is quite easy to do, noting that

$$\left(\frac{\partial \Sigma(r,\theta;\theta_0)}{\partial y}\right)_{\substack{y=0\\x>0}} + \left(\frac{\partial \Sigma(r,\theta;4\pi-\theta_0)}{\partial y}\right)_{\substack{y=0\\x>0}} = 0\,, \tag{28}$$

from which we can derive that

$$u = \Sigma(r,\theta;\theta_0) + \Sigma(r,\theta;4\pi-\theta_0)\,, \tag{29}$$

which presents two of the properties required for solving the Sommerfeld /III,28 problem: This combination proves the Helmholtz equation, and its normal derivative vanishes on the screen. It then remains to investigate its behavior at infinity. This is given by

$$u \sim \frac{1}{2} e^{ikr\cos(\theta-\theta_0)}\left(1+\operatorname{sgn}\cos\frac{\theta-\theta_0}{2}\right) + \frac{1}{2} e^{ikr\cos(\theta+\theta_0)}\left(1+\operatorname{sgn}\cos\frac{\theta+\theta_0}{2}\right) \tag{30}$$

where we agree that

$$\operatorname{sgn} z = \begin{cases} 1 & \text{if } z>0\,, \\ -1 & \text{if } z<0\,. \end{cases} \tag{31}$$

However, it is necessary to establish a convention as to the variational domain of θ. We will, therefore, assume that

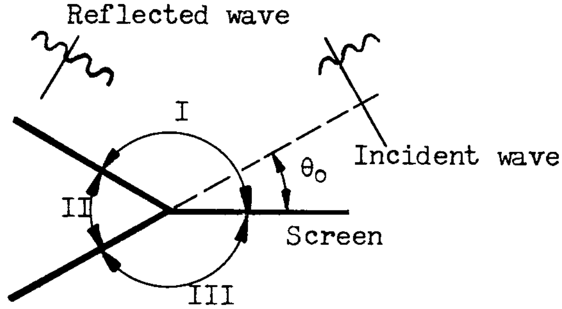
$$0 \leq \theta \leq 2n \quad , \qquad 0 < \theta_0 < n \quad . \tag{32}$$

Using this convention, we obtain

$$\left\{ \begin{array}{lll} 0 < \theta < \theta_0 + n \; , & \frac{1}{2}\left(1 + \mathrm{sgn} \cos \frac{\theta - \theta_0}{2}\right) = 1 \; , \\ \theta_0 + n < \theta < \theta_0 + 2n \; , & \frac{1}{2}\left(1 + \mathrm{sgn} \cos \frac{\theta - \theta_0}{2}\right) = 0 \; , \\ 0 < \theta < n - \theta_0 \; , & \frac{1}{2}\left(1 + \mathrm{sgn} \cos \left(\frac{\theta + \theta_0}{2}\right)\right) = 1 \; , \\ n - \theta_0 < \theta < 2n \; , & \frac{1}{2}\left(1 + \mathrm{sgn} \cos \frac{\theta + \theta_0}{2}\right) = 0 \; . \end{array} \right. \tag{33}$$

Let us pose

$$\left\{ \begin{array}{lll} \overline{\omega}_i = e^{ikr \cos(\theta - \theta_0)} \; , & & \text{incident wave ,} \\ \overline{\omega}_r = e^{ikr \cos(\theta + \theta_0)} \; , & & \text{reflected wave .} \end{array} \right. \tag{34}$$



This will yield, for $r = \infty$:

Sector I:	$0 < \theta < n - \theta_0$	$u \sim \overline{\omega}_i + \overline{\omega}_r$
Sector II:	$n - \theta_0 < \theta < n + \theta_0$	$u \sim \overline{\omega}_i$
Sector III:	$n + \theta_0 < \theta < 2n$	$u \sim 0$

Thus, in Sector I, the acoustic field is composed of the incident wave and of the reflected wave - using this term in its most commonplace sense; in Sector II, only the incident wave is involved while Sector III is a zone of silence. We repeat that here it is a question of the behavior for very large r at $kr \gg \phi$, so that the solution will coincide with the scheme of geometric acoustics. It

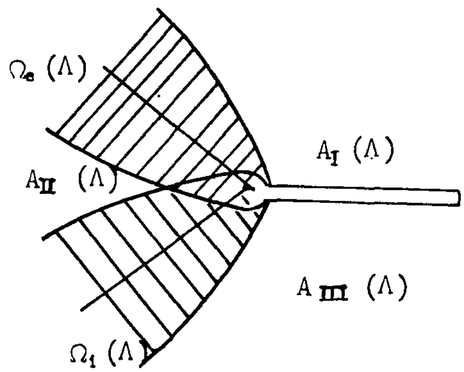
is obvious that this represents a suitable mathematical image for the interaction of a sound wave with a reflecting half-plane. /III,29

Let us now investigate, in some detail, the structure of this solution. First, in the vicinity of r = 0, the distinction into sectors is fully established, yielding

$$u \simeq 1+\sqrt{2kr}\ \frac{e^{\frac{i\pi}{4}}}{\sqrt{\pi}}\left(\cos\frac{\theta-\theta_0}{2}+\cos\frac{\theta+\theta_0}{2}\right)+\frac{ikr}{2}\Big(\cos(\theta-\theta_0)+\cos(\theta+\theta_0)\Big)+$$
$$+\,i\frac{\sqrt{2}}{3\sqrt{\pi}}\ e^{\frac{i\pi}{4}}(kr)^{3/2}\left(\cos\frac{3(\theta-\theta_0)}{2}+\cos\frac{3(\theta+\theta_0)}{2}\right)+o\big((kr)^2\big)\ . \tag{35}$$

It is readily proved that the sum of the above terms is a solution of $\Delta u \neq 0$. Thus, up to $O((kr)^2)$ exclusive, the acoustic field, in the vicinity of the boundary, is a hydrodynamic field.

Problem 13: Continue the series to the term in $(kr)^2$ and demonstrate that this term proves $\Delta u + k^2 = 0$. Write the Helmholtz equation and the limiting conditions on the screen and at infinity, in variables kr and θ, and then demonstrate that k does not intervene. Interpret this.



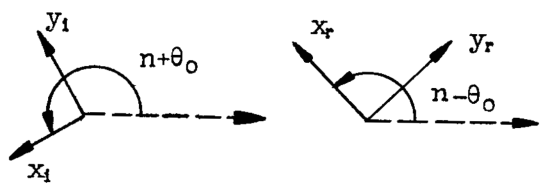
Let us now study the behavior of the solution for large r, but going beyond the stage of geometric acoustics. For this, it is necessary to plot the curves along which we have

$$\sqrt{2kr}\ \cos\frac{\theta\pm\theta_0}{2}=\pm\Lambda=\text{Const}\,. \tag{36}$$

These curves are parabolas, having the half-lines Λ_1 and Δ_r as axes, where $\overline{\omega}_1$ and $\overline{\omega}_r$ vanish discontinuously in accordance with geometric acoustics. At the

interior of these parabolas, $2kr\cos\left(\frac{\theta \pm \theta_0}{2}\right)$ varies between $-\Lambda$ and $+\Lambda$ in such a manner that, at the exterior, F is very close to 1, i.e., very close to zero, if it is assumed that Λ is very large, which we are doing here.

/III,30



Problem 14: Take the axes indicated on the accompanying diagram and demonstrate that the curve (36) admits, as equation, the following:

$$\left(k\frac{y_{i,r}}{\Lambda}\right)^2 + 2k\,x_{i,r} = \Lambda^2. \tag{37}$$

Problem 15: By writing

$$\int_{-\infty}^{\lambda} e^{-iu^2}\,du = \frac{i}{2}\int_{-\infty}^{\lambda} u^{-1}\,d(e^{-iu^2}), \tag{38}$$

demonstrate that

$$\mathcal{F}(\pm\Lambda) = \frac{1}{2}(1+\operatorname{sgn}\Lambda) \pm \frac{e^{\frac{3\pi i}{4}}}{2\sqrt{\pi}}\,\frac{e^{-i\Lambda^2}}{\Lambda}\sum_{n=0}^{N}\left(\frac{i}{2\Lambda^2}\right)^n + \frac{e^{\frac{3\pi i}{4}}}{2\sqrt{\pi}}\left(\frac{i}{2}\right)^N\int_{\pm\infty}^{\pm\Lambda}\frac{e^{-iu^2}}{u^{2N+2}}\,du \tag{39}$$

and show that the last term is augmented by const $\Lambda^{-2(N+2)}2^{-N}$.

Denoting by $\Omega_{n,i}(\Lambda)$ the domains interior to the above parabolas and by $A_I(\Lambda)$, $A_{II}(\Lambda)$, $A_{III}(\Lambda)$ the regions of Sectors I, II, and III exterior to these parabolas, we will obtain

$$\left\{\begin{array}{ll} A_I(\Lambda): & u \sim \bar{\omega}_i + \bar{\omega}_R + O\left(\frac{1}{\Lambda}\right), \\ \Omega_R(\Lambda): & \bar{\omega}_R \text{ fades progressively, going from one edge to the other,} \\ A_{II}(\Lambda): & u \sim \bar{\omega}_i + O\left(\frac{1}{\Lambda}\right), \\ \Omega_i(\Lambda): & \bar{\omega}_i \text{ fades progressively going from one edge to the other,} \\ A_{III}(\Lambda): & u \sim O\left(\frac{1}{\Lambda}\right). \end{array}\right. \tag{40}$$

It will be seen that, passing to infinity along a ray located in one of the sectors, one will finally end up by leaving the entire domain of the type $\Omega(\Lambda)$.

3.2.3 Significance of the Sommerfeld Problem

It is natural to raise the question as to the physical range of the Sommerfeld problem. A half-plane screen does not exist in nature and merely represents a mathematical model. Such a schematization is useful for studying, /III,31 on the basis of geometric acoustics, the passage of a plane wave through an aperture of large dimensions. Here, geometric optics no longer is a useful tool in the vicinity of the aperture rim. Let $\mathfrak{R}$ be the radius of curvature of the curve limiting this rim and let R_1 and R_2 be the principal radii of curvature of the screen surface. If KR, KR_1, and KR_2 are all large, it is more or less obvious that a first approximation of the acoustic field will be obtained in the neighborhood of each point of the edge, by solving a local Sommerfeld problem. Unfortunately, in acoustics, the above conditions are hardly ever fulfilled, in opposition to what happens in optics. However, it must be emphasized that the solution of the aperture, using Sommerfeld's method, is of exceptional elegance.

Problem 15: Posing

$$K^2(\xi+i\eta)^2 = x+iy, \tag{41}$$

demonstrate that, if the incident wave is parallel to the screen, the solution of the Sommerfeld problem will be

$$u = \frac{1}{2}\left(e^{iKy} + e^{-iKy}\right) + \frac{e^{i\pi/4}}{\sqrt{\pi}}\left\{ e^{iKy}\int_0^{\xi+\eta} e^{-i\lambda^2}\,d\lambda + e^{-iKy}\int_0^{\xi-\eta} e^{-i\lambda^2}\,d\lambda \right\}. \tag{42}$$

Problem 16: Posing

$$u = e^{\pm iKy}\, U^{(+,-)}(\xi,\eta), \tag{43}$$

demonstrate that, if u proves the Helmholtz equation, then $U^{\pm}$ will also prove

$$\frac{\partial^2 U^{(+,-)}}{\partial \xi^2} + \frac{\partial^2 U^{(+,-)}}{\partial \eta^2} \pm 4i\left(\eta\frac{\partial U^{(+,-)}}{\partial \xi} + \xi\frac{\partial U^{(+,-)}}{\partial \eta}\right) = 0, \tag{44}$$

and that, consequently, a possible form will be the following:

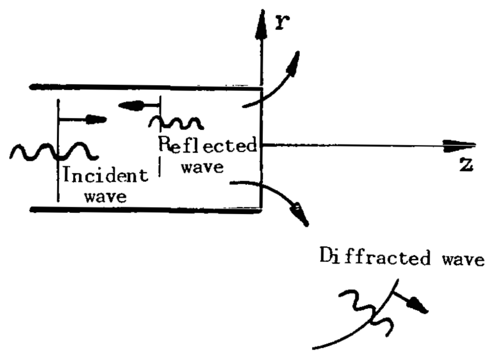
$$U^{(+,-)} = A^{(+,-)} + B^{(+,-)} \int_0^{\xi \pm \eta} e^{-i\lambda^2} d\lambda . \tag{45}$$

Pick up the result of the preceding problem.

3.3 Reflection at the Open End of a Circular Pipe /III,32

3.3.1 Formulation of the Problem

It is assumed that the tube has a diameter 2a and that its extremity $z = 0$ is open to the ambient air. The aim is to investigate whether a configuration



of stationary waves of pulsation ω exists, with a wave number $k = \frac{\omega}{c}$, for which, at infinity upstream in the tube, a double sinusoidal wave train of the following form is present:

$$u \sim e^{-ikz} + \mathcal{R} e^{ikz} . \qquad z \to -\infty \qquad 0 < r < a . \tag{1}$$

It is obvious, on re-establishing the time dependence in the form of

$$u e^{i\omega t} \sim e^{-ik(z-ct)} - \mathcal{R} e^{ik(z+ct)} \tag{2}$$

that the term e^{-ikt} represents a plane Helmholtz wave, traveling toward the open end, while the term e^{ikt} represents a plane Helmholtz wave traveling in the opposite direction and having an amplitude $|\mathfrak{R}|$ relative to that of the incident wave. Thus, the coefficient $\mathfrak{R}$ which, a priori, is complex and which can be written as follows:

$$\mathfrak{R} = |\mathfrak{R}| e^{-2ik\ell}, \tag{3}$$

can be called the complex reflection coefficient of the extremity. If $e^{-ik(z-ct)}$ is the incident wave train, we will have, by reflection, $-|\mathfrak{R}|\ e^{ik(z+ct-2\ell)}$, where the amplitude variation in the ratio $|\mathfrak{R}|:1$ is accompanied by a change of sign and by a divergence of the origin. If $ka \ll 1$, i.e., if the wavelength $\lambda = \frac{2\pi}{k} \gg a$, it becomes necessary to return to the sectional flow approximation namely,

$$\lim_{ka \to 0} |\mathfrak{R}| = 1 \quad , \lim_{ka \to 0} \ell = 0 \quad , \tag{4}$$

i.e., a reflection without divergence but with change of sign. Actually, a minor deviation of the order of a is present since, in terms of this investigation, we find that /III,33

$$\lim_{ka \to 0} \frac{\ell}{a} = 0.6133 . \tag{5}$$

In the general case, $|\mathfrak{R}|$ and $\frac{\ell}{a}$ are decreasing functions of ka. Thus, $|\mathfrak{R}|$ is less than 1, and only a part of the sound energy of the incident wave train is contained in the reflected train, while the remainder escapes through the open end into the ambient space. For a system of diffracted sound waves, it is necessary to impose a radiation condition. To express this precisely, if $|x| = \sqrt{r^2 + z^2}$ and $x = |x|\ \omega$, we know that - for large values of $|x|$ - we will have

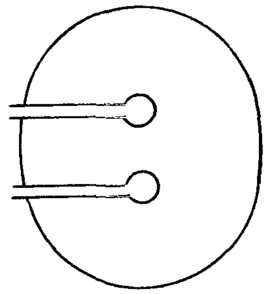
$$u_d \cong I(\omega) \frac{e^{-ik\{|x| + \Delta(\omega)\}}}{4\pi |x|}, \tag{6}$$

in such a manner that $u_d e^{i\omega t}$ represents, in the direction ω, a plane wave train traveling in the direction ω and having an amplitude (locally) of $\frac{I(\omega)}{4\pi|x|}$. The functions $I(\omega)$ and $\Delta(\omega)$ are unknown and must be obtained during the investigation. Conversely, it is now possible to state that

$$\pi a^2 |\mathfrak{R}|^2 + \iint_{|\omega|=1} \frac{I^2(\omega)}{4\pi} d\omega = \pi a^2 , \tag{7}$$

as a consequence of the law of the conservation of energy.

Problem 17: Apply the energy identity to the contour of the accompanying diagram and derive from this the relation (7). This contour is the meridian of



a surface of revolution. Should a condition be imposed on the rim of the tube?

Problem 18: Demonstrate that, at a distance, the surface density vector of the energy flux, associated with u_d, is radial and that its modulus has a value of $\frac{\rho\, ck^2}{2\pi}\,\frac{I^2(\omega)}{|x|^2}$. Derive from this that the power radiated by /III,34
the extreme end of the tube will be

$$\frac{\rho c k^2}{2}\iint_{|\omega|} I^2(\omega)\, d\omega. \tag{8}$$

To define the problem, it is necessary to impose a condition on the wall of the tube, which obviously will be

$$\frac{du}{dr}=0 \quad , \qquad r=a. \tag{9}$$

Finally, it is necessary to define the kind of behavior of u in the vicinity of $t = 0$, $r = a$. The answer to this question is obtained by studying the Sommerfeld problem, according to which one must set $u = \text{const} + O(\rho^{1/2})$, denoting by ρ the distance to the edge. In addition, it is sufficient to stipulate that the limit of the integral on a circle of radius ρ, surrounding a point of the flux boundary of ∇u_d, will tend to zero as $\rho \to 0$, which would mean that the boundary itself does not radiate

$$\lim_{\rho\to 0} \rho\,|\nabla u_d| = 0. \tag{10}$$

Formal statement of the problem: Find a solution u_d of the Helmholtz equation, outside of $z \leq 0$, $r = 0$, such that

$$
\left\{
\begin{array}{llr}
\dfrac{du_d}{dr} = 0 , & r = a , \; t \leq 0 & a) \\
\lim\limits_{t \to -\infty} \left\{ u_d - e^{-iKt} - R\, e^{iKt} \right\} = 0 , & 0 < r < a & b) \\
|x| \left(\dfrac{\partial u_d}{\partial |x|} + iK u_d \right) \to 0 , & |x| \to \infty & c) \\
\left\{ (r-a)^2 + t^2 \right\}^{\frac{1}{2}} |\nabla u_d| \to 0 . & \left\{ (r-a)^2 + t^2 \right\}^{\frac{1}{2}} \to 0 & d)
\end{array}
\right. \tag{11}
$$

3.3.2 Utilization of the Laplace Transformation

We will restrict our discussion to searching for a function u_d having a symmetry of revolution about the axis oz. Let us pose

$$
\hat{u}_d(r,\zeta) = \int_{-\infty}^{\infty} e^{-\zeta t}\, u_d(t,r)\, dt , \qquad \zeta = \xi + i\eta \tag{12}
$$

and introduce /III,35

$$
\left\{
\begin{array}{l}
A(\zeta) = \hat{u}_d(a+0, \zeta) - \hat{u}_d(a-0, \zeta) , \\
W(\zeta) = \dfrac{\partial \hat{u}_d(a,\zeta)}{\partial r} ,
\end{array}
\right. \tag{13}
$$

where the function $\hat{u}_d$ verifies

$$
\frac{\partial^2 \hat{u}_d}{\partial r^2} + \frac{1}{r}\frac{\partial \hat{u}_d}{\partial r} + (\zeta^2 + K^2)\hat{u}_d = 0 , \tag{14}
$$

from which we derive

$$
\hat{u}_d = \left\{
\begin{array}{lll}
\alpha & J_0\{(K^2+\zeta^2)^{1/2} r\} , & r < a \\
\beta & H_0^{(2)}\{(K^2+\zeta^2)^{1/2} r\} , & r > a
\end{array}
\right. \tag{15}
$$

where J_0 and $H_0^{(2)}$ are, respectively, Bessel functions of zero order, of the first and third kind. The choice of J_0 is imposed for $r < a$ in such a manner that $\hat{u}_d$ be regular in $r = 0$, whereas if $H_0^{(2)}$ rather than $H_0^{(1)}$ is selected, this choice is imposed because of the condition (11a).

Problem 19: Making use of the problems 1, 2, and 3 of Chapter II, demonstrate that

$$H_0^{(1,2)}(\kappa r) = \pm \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{e^{\pm i\kappa\sqrt{r^2+z^2}}}{\sqrt{r^2+z^2}}\, dz \quad , \qquad \begin{matrix} + \to 1 \\ - \to 2 \end{matrix} \tag{16}$$

and derive from this that

$$H_0^{(1,2)}\{(\kappa^2-\eta^2)^{\frac{1}{2}} r\} = \pm \frac{1}{\pi i} \int_{-\infty}^{\infty} \frac{e^{\pm i\kappa\sqrt{r^2+t^2}\, - i\eta t}}{\sqrt{r^2+t^2}}\, dt \quad , \tag{17}$$

by effecting a change in variables $z = r \sinh(\varphi - \varphi_0)$.

Problem 20: Let U_1 and U_2 be two solutions that are linearly independent of the Bessel equation:

$$x^2 \frac{d^2 U}{dx^2} + x \frac{dU}{dx} + (x^2 - \nu^2) U = 0 \quad , \tag{18}$$

and demonstrate that we have

$$\frac{d}{dx}\left\{ x \left(U_1 \frac{dU_2}{dx} - U_2 \frac{dU_1}{dx} \right) \right\} = 0 \tag{19}$$

Then, α and β are determined by application of eq.(13) and by making use of the relations /III,36

$$\begin{cases} J_0'(x) = -J_1(x) \quad , \quad H_0^{(2)}(x)' = -H_1^{(2)}(x), \\ J_1(x)\, H_0^{(2)}(x) - J_0(x)\, H_1^{(2)}(x) = \dfrac{2}{\pi i x} \quad , \end{cases} \tag{20}$$

yielding, on elimination of α and β,

$$i\pi(\kappa^2+\zeta^2)\, H_1^{(2)}\{(\kappa^2+\zeta^2)^{1/2} a\}\, J_1\{(\kappa^2+\zeta^2)^{\frac{1}{2}} a\}\, \hat{H}(\zeta) = \frac{2}{a}\, \hat{W}(\zeta) \, . \tag{21}$$

The mathematical problem to be solved here thus consists in finding two analytic functions of ζ, $\hat{H}(\zeta)$ and $\hat{W}(\zeta)$, knowing that they verify eq.(21) and that they are, respectively, Laplace transforms of functions that vanish at $z > 0$ and at $z < 0$. Let $h(z)$ and $w(z)$ be the originators of $\hat{H}$ and $\hat{W}$, yielding

$$\begin{cases} h(z)=0\,, & z>0 \\ \lim\limits_{z\to-\infty}\left\{h(z)-e^{-ikz}+R\,e^{ikz}\right\}=0\,, \end{cases} \tag{22}$$

from which it follows that

$$\hat{H}(\zeta) = -\frac{1}{\zeta+ik}+\frac{R}{\zeta-ik}+\hat{H}_0(\zeta)\,, \tag{23}$$

where $H_0(\zeta)$ is analytic for $\xi = \mathrm{Re}\,\zeta < 0$ and is bounded in modulus at $\xi \le 0$. In addition, according to eq.(11d), the quantity $|z|^{-1/2}|h(z)|$ remains bounded as $z \to 0_-$; from this, we derive

$$|\zeta^{3/2}\hat{H}(\zeta)| < \text{const}\,. \qquad (\xi<0) \tag{24}$$

In an analogous manner, $\hat{W}(\zeta)$ is analytic for $\xi > 0$ and is bounded in modulus for $\xi \ge 0$, whereas

$$|\zeta|^{1/2}|\hat{W}(\zeta)| < \text{const} \qquad \xi>0\,. \tag{25}$$

The function $L(\zeta) = \pi i\, H_1^{(2)}\{(k^2+\zeta^2)^{\frac{1}{2}}a\}\, J_1\{(k^2+\zeta^2)^{\frac{1}{2}}a\}$ obviously is analytic at any point at which $(k^2+\zeta^2)^{1/2} \neq 0$; in $\zeta = \pm ik$, this function /III,37 remains bounded but has branchings of the logarithmic type, as results from the following formula:

$$H_1^{(2)}(x)\,J_1(x) = x^2\left\{A(x^2)+\frac{2i}{\pi}\,\mathrm{Log}\left(\frac{x}{2}\right)\right\}+i\,B(x^2) \tag{26}$$

where A and B are integral functions of their arguments.

Problem 21: Establish eq.(26) by means of the following expansions in series:

$$\begin{cases} J_1(x)=\displaystyle\sum_{n=0}^{\infty}\frac{(-)^n\,(x/2)^{2n+1}}{n!\,(n+1)!} \\ H_1^{(2)}(x)= J_1(x)-i\,Y_1(x) \end{cases} \tag{27}$$

$$
\left\{
\begin{aligned}
&Y_1(x) = \frac{2}{\pi} \operatorname{Log}\left(\frac{x}{2}\right) J_1(x) - \frac{2}{\pi x} - \frac{1}{\pi} \sum_{n=0}^{\infty} (-1)^n \frac{(x/2)^{2n+1}}{n!(n+1)!} \{\psi(n+1) + \psi(n+2)\} \\
&\psi(n+1) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - C \qquad \psi_1 = -C \\
&C = 0{,}5772 = \text{Euler constant.}
\end{aligned}
\right.
$$

Let us pose

$$
\left\{
\begin{aligned}
&F^-(\zeta) = (k^2 + \zeta^2)\, \hat{H}(\zeta), \\
&F^+(\zeta) = \frac{2}{a}\, \hat{W}(\zeta), \\
&L(\zeta) = i\pi\, H_1^{(2)}\{(k^2+\zeta^2)^{1/2} a\}\, J_1\{(k^2+\zeta^2)^{1/2} a\}.
\end{aligned}
\right.
\tag{28}
$$

Then, the functions $F^+(\zeta)$, $F^-(\zeta)$ are analytic in the respective half-planes $\xi > 0$ and $\xi < 0$ and are bounded in $\xi \geq 0$ and $\xi \leq 0$. In addition, $\zeta^{-1/2} F^-(\zeta)$ and $\zeta^{1/2} F^+(\zeta)$ are bounded in modulus, respectively, in $\xi < 0$ and in $\xi > 0$. Finally, $L(\zeta)$ is an analytic function of ζ in the entire plane, except at the branching points $\zeta = \pm ik$. The problem, formulated in the preceding Section, reduces to seeking the functions $F^+(\zeta)$ and $F^-(\zeta)$ which, in addition to the preceding conditions, also satisfy the equation of the Wiener-Hopf type:

$$
F^-(i\eta)\, L(i\eta) = F^+(i\eta),
\tag{29}
$$

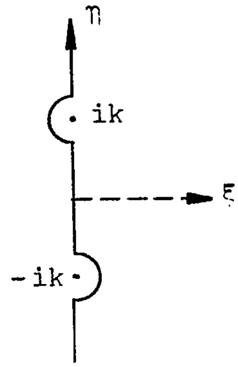
where the quantities $F^{\pm}(i\eta)$ define the respective limits of $F^+(\xi + i\eta)$ and $F^-(\xi + i\eta)$ as soon as $\xi \to 0$ for values that are, respectively, positive /III,38
and negative.

It should be mentioned that $\hat{u}$, in contrast to $\hat{H}$ and $\hat{W}$, is defined only for $\xi = 0$ and that, for this reason, we must set $\xi = 0$ in writing eq.(29); however, we will here solve this equation by the classical Wiener-Hopf method, by reasoning in the complex plane $\xi + i\eta$.

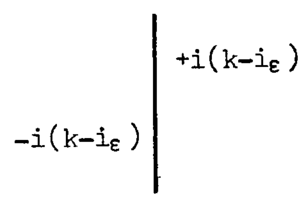
For defining $L(i\eta)$ it is necessary to select the determination of $(k^2 + \zeta^2)^{1/2}$, which will be done by adopting the following convention:

$$
\left\{
\begin{aligned}
&\operatorname{Arg}\{k^2 + (i\eta)^2\}^{\frac{1}{2}} = 0, && |\eta| < k \\
&\operatorname{Arg}\{k^2 + (i\eta)^2\}^{\frac{1}{2}} = -\pi/2, && |\eta| > k
\end{aligned}
\right.
\tag{30}
$$

which reduces to admitting that the point ζ is displaced along the imaginary axis which has two indentations, at $\pm ik$, as shown in the accompanying diagram.



In addition, our problem could be treated by passage to the limit, replacing k by $k - i\epsilon$ where $\epsilon > 0$ tends to zero so that the arrangement will be that of the accompanying diagram. The values of $\arg \{\zeta^2 + (k - i\epsilon)^2\}^{1/2}$ tend toward the



indicated values as soon as $\epsilon \to 0$. Having thus defined this convention, let us assume that the quantity $L(\zeta)$ can be written in the form of

$$L(\zeta) = \frac{L^-(\zeta)}{L^+(\zeta)}, \tag{31}$$

where, respectively, $L^+(\zeta)$ and $L^-(\zeta)$ are analytic in $\xi > 0$ and $\xi < 0$, being continuous in $\xi \geq 0$ and $\xi \leq 0$. Then, eq.(29) can be given the form

$$F^+(i\eta)\, L^+(i\eta) = F^-(i\eta)\, L^-(i\eta), \tag{32}$$

which expresses that the functions $L^+(\zeta)\, F^+(\zeta)$ and $L^-(\zeta)\, F^-(\zeta)$ that are analytic in $\xi > 0$ and in $\xi < 0$, can be extended over $\xi = 0$ and there take the same value, so that these two functions are the analytic prolongation of each other /III,39
across $\xi = 0$ and are equal at a same integral function. If we were able to demonstrate that this function remains bounded as $|\zeta| \to \infty$, the Liouville theorem would show that this integral function is equal to a constant C which then would represent the common value, independent of η, of the two sides of eq.(32).

From this, we can derive the solution of the sought problem, in the form of

$$F^{+}(\zeta) = \frac{C_1}{L^{+}(\zeta)}, \qquad F^{-}(\zeta) = \frac{C_1}{L^{-}(\zeta)}, \tag{33}$$

where the constant C is determined by comparison of eq.(33) with eqs.(23) and (28), i.e.,

$$\hat{H}(\zeta) = \frac{L^{-}(-i\kappa)}{L^{-}(\zeta)}\left\{\frac{1}{\zeta - i\kappa} - \frac{1}{\zeta + i\kappa}\right\}, \tag{34}$$

and, consequently,

$$\Re = \frac{L^{-}(-i\kappa)}{L^{-}(i\kappa)}. \tag{35}$$

Having obtained the value of the reflection coefficient $\Re$, let us now attempt to demonstrate that, at infinity and at the exterior of the tube, u_d has exactly the form of eq.(6). To achieve this, let us represent u_d in terms of h(z), by noting that

$$\Delta u_d + \kappa^2 u_d = h(z)\,\frac{\partial\,\delta(r-a)}{\partial r}. \tag{36}$$

Let $z\,e_1 + a\,\Theta$ be a point of the surface of the tube (e_1, Θ being unitary) and let $|x|\,\omega = x$ be a point in space; posing $R^* = \left|\,|x|\,\omega - z\,e_1 - r\,\Theta\right|$, we obtain

/III,40

$$u(|x|\omega) = \frac{a}{4\pi}\left\{\frac{\partial}{\partial r}\int_{-\infty}^{0} dz \int_{|\Theta|=1} d\Theta\; h(z)\,\frac{e^{-i\kappa R^*}}{R^*}\right\}_{r=a}. \tag{37}$$

Since, at $|x| \to \infty$, we have $R^* \cong |x| - z\,e_1 \cdot \omega - a\,\omega\,\Theta + O\left(\frac{1}{n}\right)$, we will obtain for the asymptotic behavior of u_d

$$u_d \cong \frac{a\,e^{-i\kappa|x|}}{4\pi\,|x|}\left\{\frac{\partial}{\partial r}\int_{-\infty}^{0} dz \int_{|\Theta|=1} d\Theta\; h(z)\; e^{i\kappa e_1\cdot\omega z + i\kappa r\,\omega\cdot\Theta}\right\}_{r=a}. \tag{38}$$

Looking for the behavior of u_d in a direction making an angle θ with the positive

axis oz, we will obtain $e_1 \cdot \omega = \cos\theta$ $\omega \cdot \Theta = \sin\theta \cdot \cos\varphi$, $\int d\Theta = \int_0^{2\pi} d\varphi$, so that, taking into consideration

$$\int_0^{2\pi} e^{ix\cos\varphi}\, d\varphi = 2\pi J_0(x) \quad , \qquad J_0'(x) = -J_1(x) \; , \tag{39}$$

we obtain

$$u_d \simeq -\left\{\frac{ka\sin\theta}{2} J_1(ka\sin\theta)\int_{-\infty}^{0} \hat{h}(z)\, e^{ikz\cos\theta}\, dt\right\} \frac{e^{-ik|x|}}{|x|} \; . \tag{40}$$

Then, according to the definition of $\hat{H}(\zeta)$ and by comparison with eq.(6), we can derive from this the values of $I(\omega)$ and $\Delta(\omega)$, namely,

$$I(\omega)\, e^{-ik\Delta(\omega)} = -\frac{2i\sqrt{\pi}\, a}{\sin\theta} J_1(ka\sin\theta)\frac{\bar{L}(-ik)}{\bar{L}(-ik\cos\theta)} = I(\theta)\, e^{-ik\Delta(\theta)} \; . \tag{41}$$

It is convenient to present the results in terms of the power radiated in the direction delimited by the angle θ, by defining

$$G(\theta) = \frac{2\, I^2(\theta)}{\int_0^{\pi} I^2(\theta)\sin\theta\, d\theta} \; , \tag{42}$$

so that the power radiated in the solid angle $d\Omega$ about the direction making an angle θ with oz, will then be

$$\frac{1}{2}\rho k c^2 \pi a^2 \left\{1 - |R|^2\right\} G(\theta)\, \frac{d\Omega}{4\pi} \; , \tag{43}$$

taking eq.(7) into consideration. This yields /III,41

$$G(\theta) = \left(\frac{J_1(ka\sin\theta)}{\sin\theta}\right)^2 \frac{4}{1-|R|^2} \left|\frac{\bar{L}(ik)}{\bar{L}(ik\cos\theta)}\right|^2 , \tag{44}$$

so that, a posteriori, it can be verified that

$$\int_0^{\pi} G(\theta) \sin\theta \, d\theta = 2. \tag{45}$$

3.3.3 Study of Factorization

We will now show that eq.(29) can effectively be brought to the form of eq.(32) with the required properties. Let us first mention that $L(i\eta)$ vanishes only if $J_1\{(k^2 - \eta^2)^{1/2}a\} = 0$ since it can be proved that $H_1^{(2)}(x)$ has no zero for $-\pi/2 \leq \arg x \leq 3\pi/2$. Since, on the other hand, it can be shown that $J_1(x)$ has a zero only for a real and positive x, it is obvious that the zeros of $J_1\{(k^2 - \eta^2)^{1/2}a\}$ have the following form:

$$\pm \eta_n = \pm\sqrt{k^2 - \gamma_n^2/a^2}, \qquad n = 1, 2, \ldots \qquad \gamma_n \leq ka \tag{46}$$

where the quantities γ_n are the zeros of $J_1(x)$. The sequence of γ_n is denumerable, yielding

$$\frac{\gamma_1}{\pi} = 1{,}2197, \qquad \frac{\gamma_2}{\pi} = 2{,}233, \qquad \frac{\gamma_3}{\pi} = 3{,}2383, \tag{47}$$

with the following asymptotic behavior:

$$\gamma_n \simeq \left(n + \frac{1}{4}\right)\pi + O\left(\frac{1}{n}\right), \tag{48}$$

when $n \to \infty$. If we thus write

$$\left\{ \begin{aligned} L(\zeta) &= \frac{\prod_{n=1}^{N} \{-(a^2\zeta^2 + a^2\eta_n^2)\}}{(1 - a^2\zeta^2)^N} \mathcal{L}(\zeta), \\ \gamma_N &\leq ka \leq \gamma_{N+1}, \end{aligned} \right. \tag{49}$$

then $\mathcal{L}(i\eta)$ will not vanish. The function $\log \mathcal{L}(\zeta)$ is regular along the contours C_+ and C_- of the accompanying diagram and behaves like $\log \frac{1}{\eta}$ + const as $|\eta| \to \infty$. In fact, the function tends toward 1, whereas, because of the following behavior /III,42

$$J_\nu(\zeta) \simeq \left(\frac{2}{\pi\zeta}\right)^{1/2} \left\{ \cos\left(\zeta - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \sum_{m=0}^{\infty} (-)^m \frac{(\nu, 2m)}{(2\zeta)^{2m}} \right. \tag{50}$$

$$
\left\{
\begin{aligned}
& \qquad - \sin\left(z - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \sum_{m=0}^{\infty} (-)^m \frac{(\nu, 2m+1)}{(2z)^{2m+1}} \Big\}, \\
& H_\nu(z) \cong \left(\frac{2}{\pi z}\right)^{\frac{1}{2}} e^{-i(z - \frac{\nu\pi}{2} - \pi/4)} \sum_{m=0}^{\infty} \frac{(\nu, m)}{(2iz)^{m}}, \\
& (\nu, m) = \frac{(4\nu^2-1)(4\nu^2-3^2)\cdots\{4\nu^2-(2m-1)^2\}}{2^{2m}\, m!},
\end{aligned}
\right.
$$

which is valid, provided that

$$ -\pi < \operatorname{Arg} z < \pi , \tag{51} $$

it is obvious that

$$ i\eta\, H_1^{(2)}(-i|\eta|)\; J_1(-i|\eta|) \sim -\frac{1}{|\eta|} . \tag{52} $$

Under these conditions, Cauchy's theorem makes it possible to write

$$ \operatorname{Log} \mathcal{L}(i\eta) = \frac{1}{2i\pi} \int_{C_+ + C_-} \frac{\operatorname{Log} \mathcal{L}(it)}{t - i\eta}\, dt . \tag{53} $$

In addition, as $|t| \to \infty$, we have

$$ \operatorname{Log} \mathcal{L}(\pm i|t|) \cong \operatorname{Log} \frac{1}{|t|} - i\pi + O\left(\frac{1}{|t|^2}\right) , \tag{54} $$

such that the integral converges at infinity.

Then, let us pose

$$
\left\{
\begin{aligned}
\mathcal{L}^+(z) &= \exp\left\{ -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\operatorname{Log} \mathcal{L}(-it)}{it + z}\, dt \right\} , \\
\mathcal{L}^-(z) &= \exp\left\{ \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\operatorname{Log} \mathcal{L}(it)}{it - z}\, dt \right\} ,
\end{aligned}
\right.
\tag{55}
$$

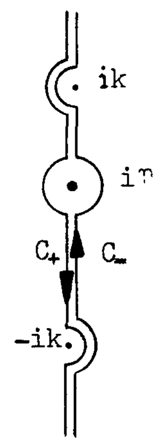
which clearly shows that

$$\frac{\mathcal{L}^-(i\eta)}{\mathcal{L}^+(i\eta)} = \mathcal{L}(i\eta), \tag{56}$$

as well as that /III.43

$$\mathcal{L}^-(\zeta) = \frac{1}{\mathcal{L}^+(-\zeta)} \tag{57}$$

and, finally, that $\mathcal{L}^+(\zeta)$ and $\mathcal{L}^-(\zeta)$ are, respectively, analytic in $\xi > 0$ and $\xi < 0$. Naturally, the integrations must comprise also the indicated indentations



about $\pm ik$, $i\eta$. Let us now form

$$\begin{cases} L^+(\zeta) = \mathcal{L}^+(\zeta)\left\{\dfrac{\prod_{n=1}^{N}\left\{-(a^2\zeta^2 + a^2\eta_n^2)\right\}^{1/2}}{(a\zeta+1)^N}\right\}^{-1}, \\ L^-(\zeta) = \mathcal{L}^-(\zeta)\,\dfrac{\prod_{n=1}^{N}\left\{-(a^2\zeta^2 + a^2\eta_n^2)\right\}^{1/2}}{(1-a\zeta)^N}. \end{cases} \tag{58}$$

On selecting a determination for the radical and retaining it, it becomes clear that $L^+(\zeta)$ and $L^-(\zeta)$ furnish the factorization posed in eq.(31).

To realize the program given in the preceding Section, it only remains to demonstrate that we have

$$\begin{cases} |L^+(\zeta)| < \text{const}\, |\zeta|^{-1/2}, & \xi \geq 0 \\ |L^-(\zeta)| < \text{const}\, |\zeta|^{1/2}, & \xi \leq 0 \end{cases} \tag{59}$$

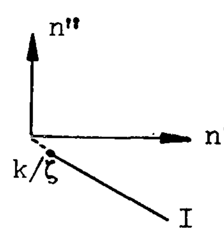
which, incidentally, is sufficient for $L^+(\zeta)$. If we write $\int_{-\infty}^{+\infty} = \int_{-k}^{k} + \int_{-\infty}^{-k} + \int_{k}^{\infty}$, then the integral $\int_{-k}^{k}$ will yield, in $L^+(\zeta)$, a contribution which remains bounded when $|\zeta| \to \infty$, $\xi \geq 0$. Thus, $\mathfrak{L}^+(\zeta)$ and thus also $L^+(\zeta)$, behave, to within a bounded multiplicative factor, like

$$\exp\left[\frac{1}{2\pi}\int_{k}^{\infty}\left\{\frac{\mathrm{Log}\,\mathfrak{L}(-it)}{it+\zeta} - \frac{\mathrm{Log}\,\mathfrak{L}(it)}{it-\zeta}\right\} dt\right]. \tag{60}$$

Now, for large values of t, the quantity $\log \mathfrak{L}(\pm it)$, because of eq.(55), behaves like $-\log t$, so that eq.(60) will be equal to the product formed by

$$\exp\left\{-\frac{\zeta}{\pi}\int_{k}^{\infty}\frac{\mathrm{Log}\, t}{t^2+\zeta^2}\right\} = O\left\{|\zeta|^{-1/2}\right\}, \tag{61}$$

with a function of ζ which remains bounded when $|\zeta| \to \infty$, $\xi \geq 0$. The /III,44
integral $\zeta \int_{k}^{\infty} \frac{\log t}{t^2 + \zeta^2} dt$ obviously is equal to $\left(\int_{k/\zeta}^{I} \frac{du}{u^2+1}\right) \log \zeta$, where $\int_{k/\zeta}^{I}$ is extended over the rectilinear path of the accompanying diagram. Finally,



as soon as $\zeta \to \infty$, we have $\int_{1/\zeta}^{1} \frac{du}{1+u^2} \to \int_{0}^{\infty} \frac{du}{1+u^2} = n/2$, which justifies eq.(61) and permits establishing eq.(59).

Taking eqs.(24) and (25) into consideration, it becomes obvious that $F^+(\zeta)L^+(\zeta) = F^-(\zeta)L^-(\zeta)$ is an integral function of ζ bounded by a constant,

which thus is a constant whose property we have utilized in the preceding Section.

3.3.4 Calculus of the Reflection Coefficient

We have immediately

$$\bar{L}(i\eta) = \left\{ \frac{(1+ia\eta)^{2N}}{\prod_{n=1}^{N} (a^2\eta^2 - a^2\gamma_n^2)} \right\}^{-1/2} \exp\left\{ \frac{1}{2\pi i} \fint_{-\infty}^{\infty} \frac{\mathrm{Log}\, \mathcal{L}(it)}{t-\eta}\, dt \right\}, \tag{62}$$

where $\fint_{-\infty}^{\infty}$ denotes the principal value of the integral. Taking $\mathcal{L}(-it) = \mathcal{L}(it)$ into consideration, we obtain

$$\fint_{-\infty}^{\infty} \frac{\mathrm{Log}\, \mathcal{L}(it)}{t-\eta}\, dt = \fint_{0}^{K} \frac{2\eta\, \mathrm{Log}\, \mathcal{L}(it)}{t^2-\eta^2}\, dt + \int_{K}^{\infty} \frac{2\eta\, \mathrm{Log}\, \mathcal{L}(it)}{t^2-\eta^2}\, dt, \tag{63}$$

and, noting that

$$\begin{cases} \mathcal{L}(it) = \dfrac{(1+a^2t^2)^N}{\prod_{n=1}^{N}(-K^2a^2+a^2t^2+\gamma_n^2)}\, \pi i\, H_1^{(2)}\{(K^2-t^2)^{\frac{1}{2}} a\} \cdot J_1\{(K^2-t^2)^{1/2} a\}, & t<K \\ \mathcal{L}(it) = \dfrac{(1+a^2t^2)^N}{\prod_{n=1}^{N}(-K^2a^2+a^2t^2+\gamma_n^2)}\, 2K_1\{(t^2-K^2)^{\frac{1}{2}} a\}\, I_1\{(t^2-K^2)^{\frac{1}{2}} a\}, & t>K \end{cases} \tag{64}$$

we find /III,45

$$\bar{L}(i\eta) = \left\{ \frac{(1+ia\eta)^{2N}}{\prod_{n=1}^{N}(-K^2a^2+a^2\eta^2+\gamma_n^2)} \right\}^{-1/2} \times$$

$$\times \exp\left\{ \frac{\eta a}{\pi i} \fint_{0}^{Ka} \mathrm{Log}\, \frac{\pi i\, H_1^{(2)}(x) J_1(x) (1+K^2a^2-x^2)^N}{\prod_{n=1}^{N}(-x^2+\gamma_n^2)} \frac{x\, dx}{(K^2a^2-a^2\eta^2+x^2)(K^2a^2+x^2)^{\frac{1}{2}}} + \right. \tag{65}$$

$$\left. + \frac{\eta a}{\pi i} \int_{0}^{\infty} \mathrm{Log}\, \frac{2K_1(x) I_1(x) (1+K^2a^2+x^2)^N}{\prod_{n=1}^{N}(\gamma_n^2+x^2)} \frac{x\, dx}{(K^2a^2-a^2\eta^2+x^2)(K^2a^2+x^2)^{1/2}} \right\}.$$

<u>Problem 22</u>: It will be recalled that

$$H_1^{(1,2)}(x) = J_1(x) \pm i Y_1(x) \qquad \begin{array}{l} + \rightarrow 1 \\ - \rightarrow 2 \end{array} \tag{66}$$

$$\left\{ \begin{array}{l} J_1(x) = x\, S(x^2) \;, \qquad S(u) = \displaystyle\sum_{n=0}^{\infty} \frac{(-)^n (u/2)^n}{n!\,(n+1)!} \;, \\ Y_1(x) = \dfrac{x}{\pi} S(x^2) \operatorname{Log} \dfrac{x}{2} - \dfrac{2}{\pi x} - \dfrac{x}{2\pi} T(x^2) \;, \\ T(u) = \displaystyle\sum_{n=0}^{\infty} (-)^n \frac{\left(\frac{u}{4}\right)^n}{n!\,(n+1)!} \left\{ \psi(n+1) + \psi(n+2) \right\} \;, \\ \psi(n) = 1 + \frac{1}{2} + \cdots + \frac{1}{n} - C \;, \quad \psi_1 = -C \;, \quad C = 0{,}5772 \;. \end{array} \right. \tag{67}$$

Demonstrate that

$$\left\{ \begin{array}{l} H_1^{(2)}(-ix) = - H_1^{(1)}(ix) = \dfrac{2}{\pi} K_1(x) \;, \\ J_1(\pm ix) = \pm \dfrac{ix}{2} S(-x^2) = \pm i I_1(x) \;; \end{array} \right. \tag{68}$$

and give the definition formulas for $I_1(x)$ and $k_1(x)$:

$$\left\{ \begin{array}{l} I_1(x) = \dfrac{x}{2} S(-x^2) \;, \\ K_1(x) = \dfrac{x}{2\pi} T(-x^2) - \dfrac{2}{\pi x} - \dfrac{1}{\pi} S(-x^2) \log\left(\dfrac{x}{2}\right) . \end{array} \right. \tag{69}$$

From this, we pass to the value of $\mathfrak{R}$

/III,46

$$\mathfrak{R} = \left(\frac{1 + iKa}{1 - iKa} \right)^N \exp\left\{ \frac{2Ka}{\pi i} \int_0^{Ka} \operatorname{Log}\left[\frac{\pi i\, H_1^{(2)}(x)\, J_1(x) (1 + K^2a^2 - x^2)^N}{\prod_{n=1}^{N} (\gamma_n^2 - x^2)} \right] \frac{dx}{x\sqrt{a^2K^2 - x^2}} \right.$$
$$\left. - \frac{2Ka}{\pi i} \int_0^{\infty} \operatorname{Log}\left[\frac{2 K_1(x)\, I_1(x) (1 + K^2a^2 + x^2)^N}{\prod_{n=1}^{N} (\gamma_n^2 + x^2)} \right] \frac{dx}{x\sqrt{a^2K^2 + x^2}} \right\} . \tag{70}$$

For calculating the modulus and the phase of the complex reflection coeffi-

cient, it should be noted that the second integral is real and thus contributes only to the phase; conversely, we have

$$\pi i \, J_1(x) \, H_1^{(2)}(x) = \pi \, J_1(x) \left\{ Y_1(x) + i J_1(x) \right\}, \tag{71}$$

from which the next theorem is obtained.

Theorem 4: Let

$$R = |R| \, \exp(2ik\ell), \tag{72}$$

be the complex reflection coefficient at the open end of a circular tube of radius a, for an incident wave $u_i = e^{-ikt}$, yielding, for the modulus,

$$|R| = \exp\left\{ -\frac{2ka}{\pi} \int_0^{ka} \frac{\operatorname{Arctan}\left(-J_1(x)/Y_1(x)\right)}{x\sqrt{k^2a^2 - x^2}} \, dx \right\} \tag{73}$$

and, for the phase,

$$\ell = N \, \frac{\operatorname{Arctg} ka}{ka} + \frac{1}{\pi} \int_0^{ka} \operatorname{Log}\left[\frac{\pi J_1(x)\sqrt{J_1^2(x) + Y_1^2(x)} \, (1 + k^2a^2 - x^2)^N}{\prod\limits_{n=1}^{N} (x^2 - \gamma_n^2)} \right] \frac{dx}{x\sqrt{k^2a^2 - x^2}} +$$
$$+ \frac{1}{\pi} \int_0^{\infty} \operatorname{Log}\left[\frac{\prod\limits_{n=1}^{N} (a^2\gamma_n^2 - k^2a^2 - x^2)}{(1 + k^2a^2 + x^2)^N \, 2K_1(x) J_1(x)} \right] \frac{dx}{x\sqrt{k^2a^2 + x^2}} . \tag{74}$$

In these formulas, γ_n denote the roots of $J_1(x) = 0$ which are inferior to ka: $\gamma_N \le ka < \gamma_{N+1}$. If $ka < 3.832$, the formula giving $\frac{\ell}{a}$ is simplified to /III,47

$$\frac{\ell}{a} = \frac{1}{\pi} \int_0^{ka} \operatorname{Log}\left\{ \pi J_1(x)\sqrt{J_1^2(x) + Y_1^2(x)} \right\} \frac{dx}{x\sqrt{a^2k^2 - x^2}} +$$
$$+ \frac{1}{\pi} \int_0^{\infty} \operatorname{Log}\left(\frac{1}{2K_1(x) I_1(x)} \right) \frac{dx}{x\sqrt{k^2a^2 + x^2}} . \tag{75}$$

Problem 23: Establish that, in the vicinity of x = 0,

$$\begin{cases} 2K_1(x)\, I_1(x) = 1 + \dfrac{x^2}{2}\left(\operatorname{Log}\dfrac{x}{2} + C' - \dfrac{1}{4}\right) + \cdots , \\ \pi J_1(x)\sqrt{J_1^2(x) + Y_1^2(x)} = 1 - \dfrac{x^2}{4}\left(\operatorname{Log}\dfrac{x}{2} + C' - \dfrac{1}{4}\right) + \cdots , \\ \operatorname{Arctan}\left(-\dfrac{J_1(x)}{Y_1(x)}\right) = \dfrac{\pi x^2}{4}\left\{1 + \dfrac{x^2}{2}\left(\operatorname{Log}\dfrac{x}{2} + C' - \dfrac{3}{4}\right) + \cdots\right\}, \end{cases} \tag{76}$$

Then, for ka $\ll$ 1, derive from this the approximating formula

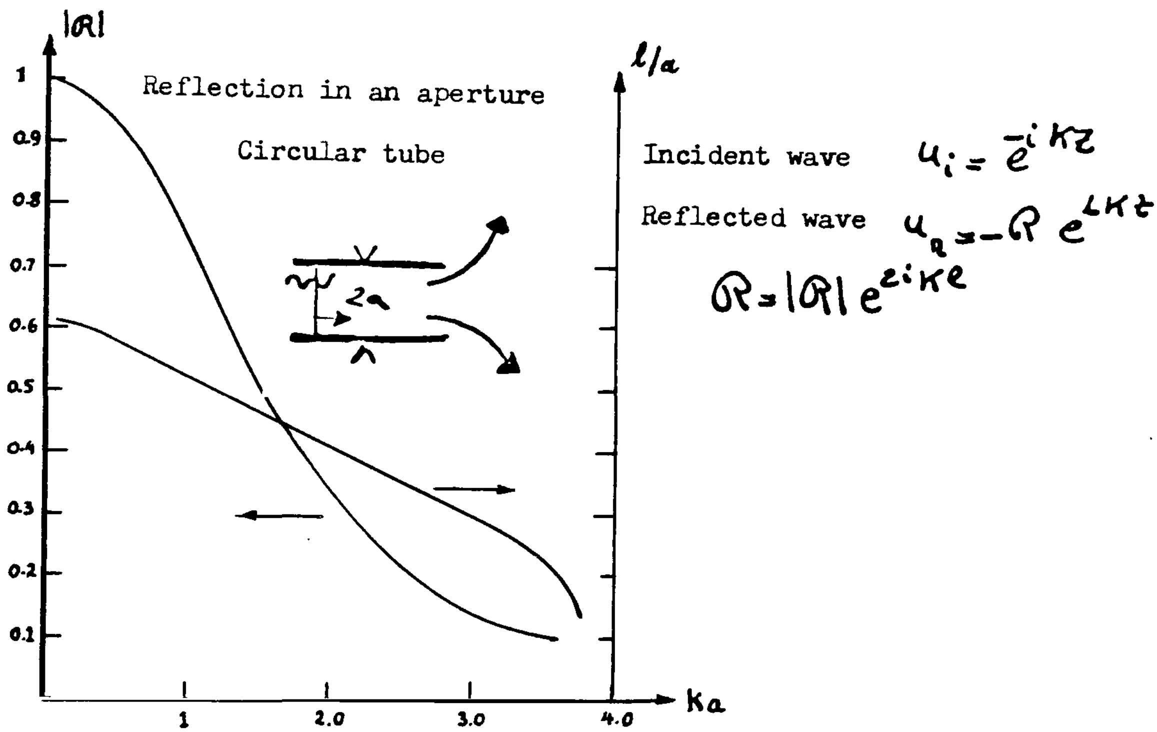
$$|R| = \exp\left(-\frac{(ka)^2}{2}\right)\left\{1 + \frac{1}{6}(ka)^4\left[\frac{13}{12} - C - \operatorname{Log}(ka)\right] + \cdots\right\} \tag{77}$$

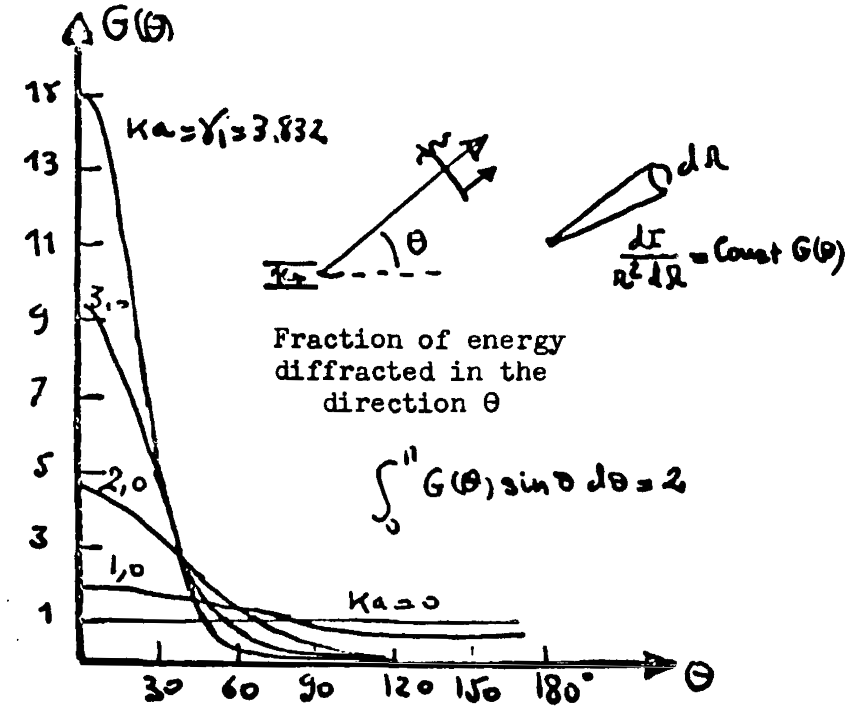
Also demonstrate that we have

$$\lim_{ka \to 0} \frac{\ell}{a} = \frac{1}{\pi}\int_0^\infty \operatorname{Log}\frac{1}{2K_1(x)J_1(x)}\,\frac{dx}{x^2} = 0.6133 . \tag{78}$$

We refer to the paper by Levine and Schwinger for more details (these authors limited themselves to ka < 3.832 but do go somewhat farther). We give some of their numerical data in the accompanying graphs.

/III,48





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CHAPTER IV

GEOMETRIC ACOUSTICS

4.1 Propagation of Sound in a Nonhomogeneous Medium in Motion

4.1.1 Equations of Acoustics

Let us return to eqs.(1) of Section 1.1 but let us equate the right-hand side of eq.(1c) to $\mathfrak{S}_0$. This situation is likely to furnish a schematization of the physical conditions prevailing in the atmosphere, where $\mathfrak{S}_0$ represents a supply of energy of exterior origin (solar radiation). In principle, chemical processes should also be taken into consideration but we do not wish to give here an accurate description of the state of the atmosphere, so that it is fully sufficient for our purposes to have a scheme which is coherent in itself. Then, let us assume ρ_0, e_0, h_0, T_0, p_0, S_0, $\mathbf{V}_0$, c_0 and let us imagine a perturbation of the following form:

$$\begin{cases} p = p_0 + \varepsilon p_1 + \varepsilon^2 p_2 + \cdots \;, \\ \cdots\cdots\cdots \\ \mathbf{V} = \mathbf{V}_0 + \varepsilon \mathbf{V}_1 + \varepsilon^2 \mathbf{V}_2 + \cdots \;; \end{cases} \tag{1}$$

Let us substitute in the equations of motion and let us equate to zero the coefficients of the various powers of ϵ. The equations corresponding to ϵ^0 are automatically verified if p_0, ..., $\mathbf{V}_0$ represents effectively the state of the atmosphere in the absence of perturbations. So far as τ and $\mathbf{q}$ are concerned, we will establish the hypothesis that $\tau_1 = \tau_2 = \ldots = 0$, $\mathbf{q}_1 = \mathbf{q}_2 \ldots = 0$. The reason for this is the fact that, in the equations of rank 0, the terms τ_0 and $\mathbf{q}_0$ have a relative significance of $O\left(\frac{\rho_0 c_0 H}{\mu_0}\right)$ where H is the length scale of the atmosphere in its entity, while τ_1, $\mathbf{q}_1$ have a relative significance /IV,2 of $O\left(\frac{\rho_0 c_0 L}{\mu_0}\right)$ where L is the length scale of the domain in which the perturbations take place. In the cases of interest here, we have $L \ll H$.

The equations, verified by the quantities of rank 1, are as follows:

$$\begin{cases} \rho_0 \dfrac{\partial \mathbf{V}_1}{\partial t} + \rho_0 \mathbf{V}_0 \cdot \nabla \mathbf{V}_1 + \nabla p_1 + \rho_1\left(\dfrac{\partial \mathbf{V}_0}{\partial t} + \mathbf{g}\right) + \left(\rho_1 \mathbf{V}_0 + \rho_0 \mathbf{V}_1\right) \cdot \nabla \mathbf{V}_0 = 0, & a) \\ \dfrac{\partial \rho_1}{\partial t} + \mathbf{V}_0 \cdot \nabla \rho_1 + \rho_0 \nabla \cdot \mathbf{V}_1 + \mathbf{V}_1 \cdot \nabla \rho_0 + \rho_1 \nabla \cdot \mathbf{V}_0 = 0 & b) \\ \dfrac{\partial S_1}{\partial t} + \mathbf{V}_0 \cdot \nabla S_1 + \mathbf{V}_1 \cdot \nabla S_0 = \left(\dfrac{\rho_1}{\rho_0} + \dfrac{T_1}{T_0}\right)\left(\dfrac{\partial S_0}{\partial t} + \mathbf{V}_0 \cdot \nabla S_0\right) & c) \end{cases} \tag{2}$$

to which we must add

$$p_1 = c_0^2 \rho_1 + g_{s_0} s_1 \, . \tag{3}$$

Let us now establish the equation of energy for the perturbations, by considering $E - \mathbb{M} \cdot \mathbb{V}_0 + C\left(\frac{V_0^2}{2} - e_0\right) = \mathfrak{E}_0$ where C, $\mathbb{M}$, and E denote the first terms of eqs.(1) of Section 1.1; this will yield

$$\begin{aligned}
&\frac{\partial}{\partial t}\left\{\rho\left(e - e_0 + \frac{(\mathbb{V}-\mathbb{V}_0)^2}{2}\right)\right\} + \nabla\cdot\left\{\rho\left(e - e_0 + \frac{(\mathbb{V}-\mathbb{V}_0)^2}{2}\right)\mathbb{V} + p(\mathbb{V}-\mathbb{V}_0)\right\} + \\
&+ \rho(\mathbb{V}-\mathbb{V}_0)(\nabla\mathbb{V}_0)\cdot(\mathbb{V}-\mathbb{V}_0) + \frac{\rho(\mathbb{V}_0-\mathbb{V})\cdot\nabla p_0}{\rho_0} + \frac{\rho_0 p - \rho p_0}{\rho_0}\nabla\cdot\mathbb{V}_0 \\
&+ \rho(\mathbb{V}-\mathbb{V}_0)\cdot\nabla e_0 = 0
\end{aligned} \tag{4}$$

by naturally making use of equations verified by the quantities of rank zero. The same equation (4) obviously is verified if the quantities with the subscript zero are exchanged for those that have no subscript; in addition, it follows that

/IV,3

$$\begin{aligned}
&\frac{\partial}{\partial t}\left\{(\rho-\rho_0)(e-e_0) + \frac{\rho+\rho_0}{2}|\mathbb{V}-\mathbb{V}_0|^2\right\} + \nabla\cdot\left\{(\rho\mathbb{V}-\rho_0\mathbb{V}_0)(e-e_0) + \frac{\rho\mathbb{V}+\rho_0\mathbb{V}_0}{2}|\mathbb{V}-\mathbb{V}_0|^2 + (p-p_0)(\mathbb{V}-\mathbb{V}_0)\right\} + \\
&+ (\mathbb{V}-\mathbb{V}_0)\cdot\left(\rho\nabla\mathbb{V}_0 + \rho_0\nabla\mathbb{V}\right)\cdot(\mathbb{V}-\mathbb{V}_0) + (\mathbb{V}-\mathbb{V}_0)\left(\frac{\rho_0\nabla p}{\rho} - \frac{\rho\nabla p_0}{\rho_0}\right) + (\rho_0 p - \rho p_0)\left(\frac{\nabla\cdot\mathbb{V}_0}{\rho_0} - \frac{\nabla\cdot\mathbb{V}}{\rho}\right) + \\
&+ (\mathbb{V}-\mathbb{V}_0)\cdot\left(\rho\nabla e_0 - \rho_0\nabla e\right) = 0 \, .
\end{aligned} \tag{5}$$

Substituting eq.(1) in eq.(5), we obtain an equation into which the quantities of rank 1 enter, i.e.,

$$\begin{aligned}
&\frac{\partial}{\partial t}\left(\rho_1 e_1 + \rho_0|\mathbb{V}_1|^2\right) + \nabla\cdot\left\{(\rho_1\mathbb{V}_0 + \rho_0\mathbb{V}_1)e_1 + \rho_0|\mathbb{V}_1|^2\mathbb{V}_0 + p_1\mathbb{V}_1\right\} + 2\mathbb{V}_1\cdot(\rho_0\nabla\mathbb{V}_0)\cdot\mathbb{V}_1 - \\
&- \mathbb{V}_1\cdot\left(\frac{2\rho_1}{\rho_0}\nabla p_0 - \nabla p_1\right) + (\rho_0 p_1 - \rho_1 p_0)\left(\frac{\rho_1}{\rho_0^2}\nabla\cdot\mathbb{V}_0 - \frac{\nabla\cdot\mathbb{V}_1}{\rho_0}\right) + \mathbb{V}_1\cdot\left(\rho_1\nabla e_0 - \rho_0\nabla e_1\right) = 0
\end{aligned} \tag{6}$$

to which we must add

$$e_1 = \frac{p_0}{\rho_0^2}\rho_1 + T_0 s_1 \quad , \qquad p_1 = c_0^2\rho_1 + g_{s_0} s_1 \, . \tag{7}$$

We will modify eq.(6) in such a manner that all derivations having to do with

quantities of rank 1 will have the form of divergence. For this, we write

$$\mathbb{V}_1 \cdot \nabla p_1 - \rho_0 \mathbb{V}_1 \cdot \nabla e_1 = \nabla \cdot (p_1 \mathbb{V}_1 - \rho_0 e_1 \mathbb{V}_1) + (\rho_0 e_1 - p_1)\, \nabla \cdot \mathbb{V}_1 + e_1 \mathbb{V}_1 \cdot \nabla \rho_0 \,,$$

and make use of

$$\nabla \cdot \mathbb{V}_1 = -\frac{1}{\rho_0}\left(\frac{\partial \rho_1}{\partial t} + \mathbb{V}_0 \cdot \nabla \rho_1\right) - \frac{\mathbb{V}_1 \cdot \nabla \rho_0}{\rho_0} - \frac{\rho_1}{\rho_0}\, \nabla \cdot \mathbb{V}_0 \,,$$

to arrive finally at

$$\begin{aligned}
&\frac{\partial}{\partial t}\left\{\rho_0 |\mathbb{V}_1|^2 + c_0^2 \frac{\rho_1^2}{\rho_0} + 2 g_{s_0} \frac{\rho_1 S_1}{\rho_0}\right\} + \nabla \cdot \left\{2 p_1 \mathbb{V}_1 + \left(\rho_0 V_1^2 + c_0^2 \frac{\rho_1^2}{\rho_0} + 2 g_{s_0} \frac{\rho_1 S_1}{\rho_0}\right)\mathbb{V}_0\right\} + \\
&+ 2\, \mathbb{V}_1 \cdot (\rho_0 \nabla \mathbb{V}_0) \cdot \mathbb{V}_1 + \left\{2 \frac{\rho_1^2}{\rho_0}\left(c_0^2 - \frac{p_0}{\rho_0}\right) + g_{s_0} \rho_1 S_1\right\} \nabla \cdot \mathbb{V}_0 + \\
&+ \frac{2 g_{s_0} S_1}{\rho_0} \mathbb{V}_1 \cdot \nabla \rho_0 + \rho_1^2 \left(\frac{\partial}{\partial t} + \mathbb{V}_0 \cdot \nabla\right)\left(\frac{1}{\rho_0}\left(\frac{p_0}{\rho_0} - c_0^2\right)\right) + \\
&+ \rho_1 S_1 \left(\frac{\partial}{\partial t} + \mathbb{V}_0 \cdot \nabla\right)\left(T_0 - \frac{2 g_{s_0}}{\rho_0}\right) + \\
&+ \left(1 - \frac{2 g_{s_0}}{\rho_0 T_0}\right) T_0 \rho_1 \left(\frac{\rho_1}{\rho_0} + \frac{T_1}{T_0}\right)\left(\frac{\partial S_0}{\partial t} + \mathbb{V}_0 \cdot \nabla S_0\right) = 0
\end{aligned} \tag{8}$$

with $\frac{D_0}{D_t} \equiv \frac{\partial}{\partial t} + \mathbb{V}_0 \cdot \nabla$. If ρ_0, ..., $\mathbb{V}_0$ are constants, it can always be arranged to have $\mathbb{V}_0 = 0$ so that eq.(8) is reduced to the equation constructed in Chapter I. By analogy, we can continue to use the terminology of the same Chapter. /IV,4

Acoustic energy density:

$$\mathcal{E} = \frac{1}{2}\left(\rho_0 V_1^2 + \frac{c_0^2 \rho_1^2}{\rho_0}\right) + g_{s_0} \rho_1 S_1 , \tag{9}$$

Surface density of the acoustic energy flux:

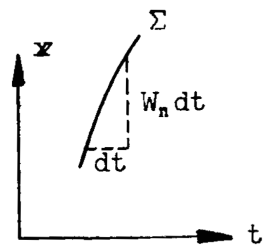
$$\mathbb{W} = p_1 \mathbb{V}_1 + \mathcal{E}\, \mathbb{V}_0 \tag{10}$$

Volume density of the acoustic energy production:

$$\Pi = \mathbf{V}_1\cdot(\rho_0 \nabla \mathbf{V}_0)\cdot \mathbf{V}_1 + \left\{ \frac{\rho_1^2}{\rho_0}\left(c_0^2 - \frac{p_0}{\rho_0}\right) + \frac{1}{2}\, g_{s_0}\, \rho_1 s_1 \right\} \nabla\cdot \mathbf{V}_0 + \frac{g_{s_0}}{\rho_0}\, s_1\, \mathbf{V}_1\cdot\nabla\rho_0 + \frac{1}{2}\rho_1^2\left(\frac{\partial}{\partial t} + \mathbf{V}_0\cdot\nabla\right)\left[\frac{1}{\rho_0}\left(\frac{p_0}{\rho_0} - c_0^2\right)\right] + \frac{1}{2}\rho_1 s_1\left(\frac{\partial}{\partial t} + \mathbf{V}_0\cdot\nabla\right)\left(T_0 - \frac{2 g_{s_0}}{\rho_0}\right) + \left(T_0 - \frac{2 g_{s_0}}{\rho_0}\right)\frac{\rho_1}{2}\left(\frac{\rho_1}{\rho_0} + \frac{T_1}{T_0}\right)\left(\frac{\partial s_0}{\partial t} + \mathbf{V}_0\cdot\nabla s_0\right) \tag{11}$$

4.1.2 <u>Characteristic Surfaces</u>

Let us consider a surface Σ whose velocity of normal displacement, vectorially, is $W\,\mathbf{n}$ where $\mathbf{n}$ is a unit vector normal to $S(t)$ directed from a certain



side, denoting by $S(t)$ the intersection of Σ by t = const. Let us pose

$$\nabla = \mathbf{n}\,\delta_n + \nabla_T\ , \tag{12}$$

so that we obtain

$$\begin{cases} \boxed{\dfrac{\partial \rho_1}{\partial t} + \mathbf{n}\cdot\mathbf{V}_0\,\delta_n \rho_1 + \rho_0\,\mathbf{n}\cdot(\delta_n \mathbf{V}_1)} + \mathbf{V}_{0T}\cdot\nabla_T\rho_1 + \rho_0\nabla_T\cdot\mathbf{V}_1 + \mathbf{V}_1\cdot\nabla\rho_0 = 0 \\ \boxed{\rho_0\dfrac{\partial \mathbf{V}_1}{\partial t} + \rho_0\,\mathbf{n}\cdot\mathbf{V}_0\,\delta_n \mathbf{V}_1 + c_0^2\,\mathbf{n}\,\delta_n\rho_1 + g_{s_0}\,\mathbf{n}\,\delta_n s_1} + \rho_0\mathbf{V}_0\cdot\nabla_T\mathbf{V}_1 + \\ \qquad + \nabla_T p_1 + \rho_1\left[\dfrac{\partial \mathbf{V}_0}{\partial t} + g\right] + (\rho_1\mathbf{V}_0 + \rho_0\mathbf{V}_1)\nabla\mathbf{V}_0 = 0 \\ \boxed{\dfrac{\partial s_1}{\partial t} + \mathbf{n}\cdot\mathbf{V}_0\,\delta_n s_1} \quad \mathbf{V}_0\cdot\nabla_T s_1 + \mathbf{V}_1\cdot\nabla s_0 = 0 \end{cases} \tag{13}$$

Here, we have boxed the terms that do not enter the derivatives $\dfrac{\partial}{\partial t}$ and δ_n <u>/IV,5</u> of the quantities of rank 1. If, across Σ, the derivatives in question undergo a discontinuity, denoted by [ ] ($\tau f \delta = f_{\text{Int. side}} - f_{\text{Ext. side}}$), the entity of these discontinuities verifies a homogeneous system

$$\left\{\begin{aligned} &\left[\frac{\partial \rho_1}{\partial t}\right] + \mathbf{n}\cdot\mathbf{V}_0\,[\delta_n \rho_1] + \rho_0\,\mathbf{n}\cdot[\delta_n \mathbf{V}_1] = 0 \;, \\ &\rho_0\left[\frac{\partial \mathbf{V}_1}{\partial t}\right] + \rho_0\,\mathbf{n}\cdot\mathbf{V}_0\,[\delta_n \mathbf{V}_1] + c_0^2\,\mathbf{n}\,[\delta_n \rho_1] + g_{s_0}\,\mathbf{n}\,[\delta_n S_1] = 0 \;, \\ &\left[\frac{\partial S_1}{\partial t}\right] + \mathbf{n}\cdot\mathbf{V}_0\,[\delta_n S_1] = 0 \;. \end{aligned}\right. \tag{14}$$

Note: For an exact solution of the equations of motion, outside of any linearization procedure and if the derivatives $\frac{\partial}{\partial t}$ and δ_n of the various quantities of flow undergo discontinuities on traversing a surface Σ, then these discontinuities will verify the same homogeneous system (14) in which the subscripts 0 and 1 have been omitted. It is important that the reader understands the mechanism which leads to this identity; similar phenomena will be discussed later.

Let us pick up the classical discussion of the system (14). Let us note that, because of the continuity of ρ_1, S_1, $\mathbf{V}_1$ themselves, we necessarily have

$$\left\{\begin{aligned} &\left[\frac{\partial \rho_1}{\partial t}\right] + W\,[\delta_n \rho_1] = 0 \;, \\ &\left[\frac{\partial \mathbf{V}_1}{\partial t}\right] + W\,[\delta_n \mathbf{V}_1] = 0 \;, \\ &\left[\frac{\partial S_1}{\partial t}\right] + W\,[\delta_n S_1] = 0 \;, \end{aligned}\right. \tag{15}$$

since $\frac{\partial}{\partial t} + W\,\delta_n$ is a differentiation that does not originate in Σ. The elimination of $\left[\frac{\partial V_1}{\partial t}\right]$ and $\left[\frac{\partial S_1}{\partial t}\right]$ between eqs.(14) and (15) will lead to

/IV,6

$$\left\{\begin{aligned} &(\mathbf{n}\cdot\mathbf{V}_0 - W)\,[\delta_n \rho_1] + \rho_0\,\mathbf{n}\cdot[\delta_n \mathbf{V}_1] = 0 \;, \\ &\rho_0(\mathbf{n}\cdot\mathbf{V}_0 - W)\,[\delta_n \mathbf{V}_1] + c_0^2\,\mathbf{n}\,[\delta_n \rho_1] + g_{s_0}\,\mathbf{n}\,[\delta_n S_1] = 0 \;, \\ &(\mathbf{n}\cdot\mathbf{V}_0 - W)\,[\delta_n S_1] = 0 \;. \end{aligned}\right. \tag{16}$$

First case: $\mathbf{n}\cdot\mathbf{V}_0 - W = 0$ (Σ shifting with the fluid)

$$
\begin{cases}
[\delta_n S_1] & \text{is arbitrary} \quad , \\
n \cdot [\delta_n V_1] = 0 \ , \\
c_0^2 [\delta_n \rho_1] + g_{s_0} [\delta_n S_1] = [\delta_n p_1] = 0 \ ,
\end{cases}
\tag{17}
$$

after which we pass to $\left[\frac{\partial S_1}{\partial t}\right]$ etc., because of eq.(15). It can be stated that Σ is an entropy wave surface. The derivatives of pressure and the normal velocity component are continuous across an entropy wave surface, while the tangential velocity and entropy components may undergo arbitrary discontinuities of normal derivatives of space with the time derivatives undergoing discontinuities linked to the preceding ones by

$$
\begin{cases}
\left[\frac{\partial V_{1\tau}}{\partial t}\right] + n \cdot V_0 [\delta_n V_{1\tau}] = 0 \ , \\
\left[\frac{\partial S_1}{\partial t}\right] + n \cdot V_0 [\delta_n S_1] = 0 .
\end{cases}
\tag{18}
$$

<u>Second case</u>: $n \cdot V_0 \neq W$.

In this case, we first have

$$
[\delta_n S_1] = 0 \ , \tag{19}
$$

by means of which the system (16) is reduced to

$$
\begin{cases}
(n \cdot V_0 - W)[\delta_n \rho_1] + \rho_0 \, n \cdot [\delta_n V_1] = 0 \ , \\
c_0^2 \, n [\delta_n \rho_1] + \rho_0 (n \cdot V_0 - W)[\delta_n V_1] = 0 .
\end{cases}
\tag{20}
$$

To have this system admit a nonidentical 0-solution, it is necessary and sufficient that the condition

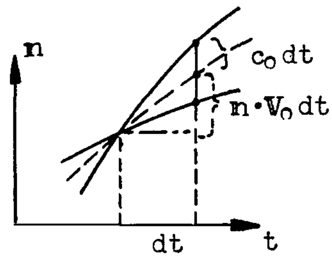
/IV,7

$$
(n \cdot V_0 - W)^2 = c_0^2 \ , \tag{21}
$$

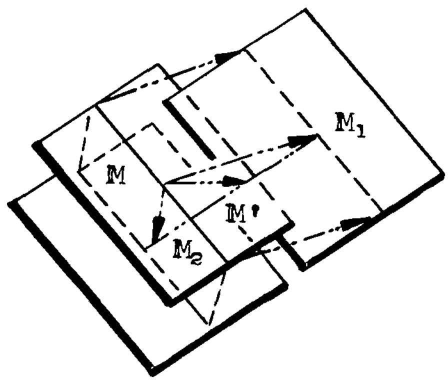
is satisfied; this condition, if we write

$$
W = n \cdot V_0 \pm c_0 \ , \tag{22}
$$

expresses that the surface S(t), known as the acoustic wave surface, is displaced at sound speed, relative to the tangent entropy wave surface, in one or the other direction of the normal.



The accompanying diagrams illustrate this propagation phenomenon. To understand the second scheme, note that $MM' = V_0 dt$, $M'M_1 = -M'M_2 = c_0 n$. The directions MM_1 and MM_2 define the tangents to the sound rays, which we will discuss later. Let us now state the results with respect to the discontinuities of the derivatives: Across a surface which represents an acoustic wave surface, i.e., a surface whose velocity of normal displacement is sonic relative to the



velocity of normal displacement of an entropy wave surface, the derivatives of the entropy are continuous; the normal derivative of space of the specific mass may suffer an arbitrary discontinuity $[\delta_n \rho_1]$, so that the normal derivative of the pressure will also suffer a discontinuity

$$[\delta_n p_1] = c_0^2 [\delta_n \rho_1]. \tag{23}$$

The time derivatives of ρ_1 and p_1 also undergo discontinuities

$$\left[\frac{\partial \rho_1}{\partial t}\right] = -n \cdot V_0 [\delta_n \rho_1], \qquad \frac{\partial p_1}{\partial t} = -n \cdot V_0 [\delta_n p_1]; \tag{24}$$

The derivatives of the tangential velocity components are continuous, while the normal velocity component undergoes discontinuities in its normal space-time derivatives, given by /IV,8

$$\begin{cases} [\delta_n V_{1n}] = \varepsilon \frac{c_0}{\rho_0} [\delta_n \rho_1], \\ \left[\frac{\partial V_{1n}}{\partial t}\right] = - \, n \cdot V_0 \, \frac{c_0}{\rho_0} [\delta_n \rho_1], \end{cases} \tag{25}$$

where $\epsilon = \pm 1$ and where the sign is so selected that the velocity of normal displacement of the wave surface will be

$$W = n \left(n \cdot V_0 + \varepsilon \, c_0 \right). \tag{26}$$

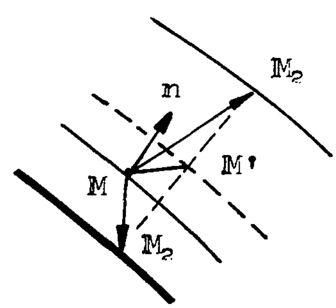
Characteristic space-time surfaces are known as three-dimensional hypersurfaces generated by the propagation of wave surfaces in ordinary space.

4.1.3 Geometric Aspects of the Propagation of Acoustic Wave Surfaces

The accompanying diagram depicts the kinematics of the propagation phenomenon, with

$$MM' = V_0 \, dt, \quad M'M_1 = - M'M_2 = c_0 \, n \, dt.$$

We will center our attention on a certain wave whose propagation we will investigate; for this, we select the vector n in the direction of propagation.



Under these conditions, we have

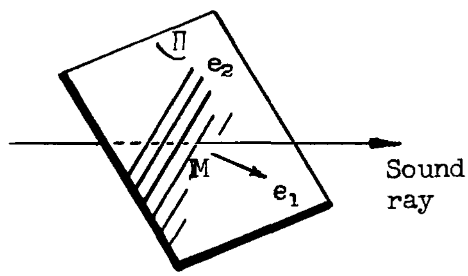
$$W = n \cdot V_0 + c_0 . \tag{27}$$

Sound ray. This ray is defined by

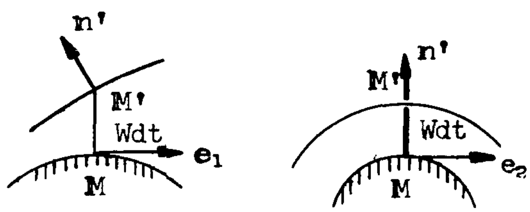
$$\frac{dM}{dt} = V_0 + c_0 n \ , \qquad (28)$$

which, in other words, means that it is the locus of the point obtained by integration of the construction which permits passing from the point M to the point M_1 in the preceding diagram. /IV,9

Bicharacteristic of the propagation ray. This is the space-time curve whose sound ray is the projection of space.



Let us return to determining the sound ray by integration of eq.(28). Here, V_0 and c_0 are functions like those of t and M, while n is not known and therefore requires a separate equation.



At the point M through which passes the sound ray in question, let us trace the plane (Π) tangent to the wave surface. To each point of this plane, a value of W can be attached since V_0 and c_0 are defined at each point and since n is constant in (Π) if, by convention, this quantity is given the value that it had in M. Let us plot, in (Π), the iso-w curves and let e_1 be a unit vector orthogonal to these curves, directed toward increasing W; e_2 will be selected orthogonal to e_1, i.e., tangent to the iso-w curves in such a manner that $n = e_1 \wedge e_2$. Let us consider two adjacent positions, corresponding to the instants t_0 and $t_0 + dt$ of the wave surface; if we make a cut through a plane normal to (Π) passing through e_2, we obtain two adjacent parallel curves; conversely, if we cut through a plane normal to (Π) passing through e_1, we will obtain two adjacent curves that are not parallel. Let n' be the unit vector normal to the position of the wave in $t + dt$ at the point M' on the normal to (Π) in M, so

that we obtain

$$\mathbb{n}' = \mathbb{n} - \mathbb{e}_1 \frac{dw}{d\nu} dt, \tag{29}$$

denoting by $\mathbb{e}_1 \frac{dw}{d\nu}$ the gradient of W in the plane (Π). It is not neces- /IV,10

sary to take into consideration the tangential displacement since, during this displacement, the direction of the vector $\mathbb{n}$ does not change. Let us define this more closely by a calculation: Let x_1 and x_2 be orthogonal curvilinear coordinates on the wave surface at the instant t_0 and let us suppose that $x_1 = x_2 = 0$ corresponds to the previously considered point $\mathbb{M}$ where the curves x_1 = const are iso-w curves. We denote by $\mathbb{M}(x_1, x_2, t)$ the successive positions of a point of the wave surface, which is followed over its propagation in such a manner that the propagation ray will be obtained by making x_1 and x_2 constant. This yields

$$\mathbb{n} = K \frac{\partial \mathbb{M}}{\partial x_1} \wedge \frac{\partial \mathbb{M}}{\partial x_2} \tag{30}$$

and, consequently,

$$\frac{d\mathbb{n}}{dt} = K\left\{\frac{\partial}{\partial x_1}(\mathbb{V}_0 + c_0 \mathbb{n}) \wedge \frac{\partial \mathbb{M}}{\partial x_2} + \frac{\partial \mathbb{M}}{\partial x_1} \wedge \frac{\partial}{\partial x_2}(\mathbb{V}_0 + c_0 \mathbb{n})\right\}_T, \tag{31}$$

where the subscript T means that projection onto the tangent plane (Π) must be made. With respect to $x_1 = x_2 = 0$, $t = t_0$, it is possible to arrange the calculation such that $\frac{\partial \mathbb{M}}{\partial x_1} = \mathbb{e}_1$, $\frac{\partial \mathbb{M}}{\partial x_2} = \mathbb{e}_2$, $K = 1$, thus yielding

$$\frac{d\mathbb{n}}{dt} = \left(\frac{\partial c_0}{\partial x_1} + \mathbb{n} \cdot \frac{\partial \mathbb{V}_0}{\partial x_1}\right) \mathbb{n} \wedge \mathbb{e}_2 + \left(\frac{\partial c_0}{\partial x_2} + \mathbb{n} \cdot \frac{\partial \mathbb{V}_0}{\partial x_2}\right) \mathbb{e}_1 \wedge \mathbb{n} \tag{32}$$

whereas, by construction, we have

$$\begin{cases} \dfrac{\partial c_0}{\partial x_1} + \mathbb{n} \cdot \dfrac{\partial \mathbb{V}_0}{\partial x_1} = \dfrac{\partial w_n}{\partial x_1}, \\ \dfrac{\partial c_0}{\partial x_2} + \mathbb{n} \cdot \dfrac{\partial \mathbb{V}_0}{\partial x_2} = \dfrac{\partial w_n}{\partial x_2}, \end{cases} \tag{33}$$

denoting by w_n the value of $w = c_0 + \mathbb{n} \cdot \mathbb{V}_0$ in the plane (Π), with $\mathbb{n}$ remaining constant there. Depending on the choice of coordinates, we have $\frac{\partial w_n}{\partial x_2} = 0$ so that eq.(32) is obtained in the form of

$$\frac{d\mathbf{n}}{dt} = - \nabla_\pi W_n \ , \tag{34}$$

which is nothing else but eq.(29).

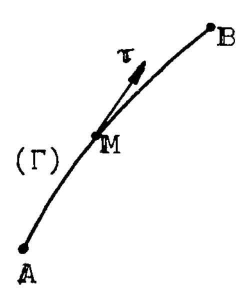
Theorem 1: Let us consider a propagation ray (C) assumed to be passed along the propagation of an acoustic wave surface and, as a consequence, parametrized as a function of time in the form of $\mathbf{M} = \mathbf{M}(t)$; to each instant, let us associate the unit vector $\mathbf{n}(t)$ normal to the wave surface in $\mathbf{M}(t)$, pointing in the direction of propagation. In $\mathbf{M}$, let us trace the plane (Π) normal to $\mathbf{n}$ and let us consider $W_n = c_0 + \mathbf{n} \cdot \mathbf{V}_0$ as a function of the position in (Π), with $\mathbf{n}$ remaining fixed. We then have /IV,11

$$\frac{d\mathbf{n}}{dt} = - \nabla_\pi W_n \ , \tag{35}$$

where ∇_T denotes the gradient vector in the plane Π. In other words, the velocity vector of the point $\mathbf{n}(t)$ is parallel to (Π), normal to the iso-w_n curves in (Π) directed toward decreasing W_n and equal, in modulus, to the gradient of W_n. The sound rays thus can be obtained by integration of the following differential system:

$$\left\{ \begin{array}{ll} \dfrac{d\mathbf{M}}{dt} = \mathbf{V}_0 + c_0 \mathbf{n} \ , & a) \\[2ex] \dfrac{d\mathbf{n}}{dt} = - \nabla_\pi c_0 - (\nabla_\pi \mathbf{V}_0) \cdot \mathbf{n} \ . & b) \end{array} \right. \tag{36}$$

Let us next investigate the specific case in which $\mathbf{V}_0$ and c_0 do not depend on time. Let us consider a curve (Γ) joining two points $\mathbf{A}$ and $\mathbf{B}$ and let $\boldsymbol{\tau}$ be

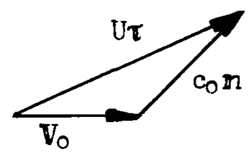


the unit vector tangent to this curve at one of its points, in the direction which runs from $\mathbf{A}$ to $\mathbf{B}$; then, independently of any consideration of propagation

by waves, we can associate a unit vector $\mathbf{n}_\Gamma$ to $\boldsymbol{\tau}$, by means of the condition

$$\mathbf{V}_0 + c_0 \mathbf{n}_\Gamma = U \boldsymbol{\tau}, \tag{37}$$

where the vector $\mathbf{n}_\Gamma$ is thus determined in a unique manner if $V_0 < c_0$ and /IV.12
if the condition $U > 0$ is imposed, representing hypotheses which will be retained



in what follows. At each point, we can thus define

$$W_\Gamma = \mathbf{V}_0 \cdot \mathbf{n}_\Gamma + c_0 , \tag{38}$$

and, consequently, also the time required for going from $\mathbb{A}$ to $\mathbb{B}$ along (Γ), in accordance with the laws of the propagation of sound

$$T(\mathbb{A}, \mathbb{B}; \Gamma) = \int_{\mathbb{A}}^{\mathbb{B}} \frac{\boldsymbol{\tau} \cdot \mathbf{n}_\Gamma}{W_\Gamma} \, d\sigma, \tag{39}$$

where dr denotes the element of arc of Γ.

Theorem 2 (Fermat principle): In a nonhomogeneous atmosphere (in a time-independent nonhomogeneous atmosphere), the sound rays (C) may be defined as those that render the time of transit from one point to the other stationary, where the time in question is defined by eq.(39). The stationary value

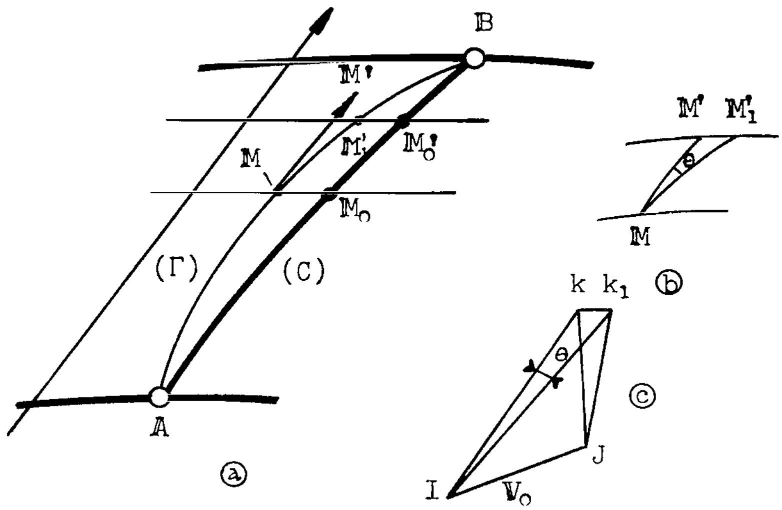
$$T(\mathbb{A}, \mathbb{B}) = \int_{\mathbb{A}C\mathbb{B}} \frac{\boldsymbol{\tau} \cdot \mathbf{n}}{W} \, d\sigma, \tag{40}$$

is the time elapsed between the passage of one wave surface in $\mathbb{A}$ and $\mathbb{B}$; this time is the same for an entire (class of) wave surface (or surfaces) which passes (or pass) in $\mathbb{A}$.

It is possible to prove this theorem without calculation, by operating as follows: Let (C) be a sound ray joining $\mathbb{A}$ and $\mathbb{B}$ and let us consider the succes-

sive positions of a wave surface passing through A and B at the respective instants t_0 and $t_0 + T(A, B)$. We will demonstrate that the variation in /IV,13

$\int_A^B \frac{d\sigma}{U}$ is of the second order when passing from the arc $A\,C\,B$ to the arc $A\,\Gamma\,B$, where the infinitely small principal is the transverse deviation of the arcs (C) and (Γ). Let us use t as common parameter for the arcs (C) and (Γ), by proceeding in the following manner: Let us trace the successive positions of the wave



surface between t_0 and $t_0 + T$ and take t as common parameter of the points M_0 and M' where the position of the wave surface, at the instant t, intersects the arcs (C) and (Γ). Let us consider the sound ray passing through M; this ray cuts the wave surface in t + dt at the point M' as soon as (Γ) intersects this surface in M_1'. For the element of arc $M\,M'$ as well as for M_0M_0', we have $\frac{d\sigma}{U} =$ $= dt$, so that it is necessary to evaluate $\frac{d\sigma}{U}$ for the element of arc $M\,M_1'$ as a function of the angle θ of the directions $M\,M'$ and $M\,M_1'$. Let us note that θ is of the same order as the infinitely small principal which had been discussed above; we then will have proved Fermat's theorem if we establish the relation $\frac{d\sigma}{U} = (1 + O(\theta^2))dt$ for the element $M\,M_1'$. A glance at the accompanying diagram, section ⓑ, with $I\,K = U\,\tau$ and $I\,K_1 = U_1\tau_1$ can be used in place of proof, if one notes that $K\,K_1$ is parallel to $M'M_1'$.

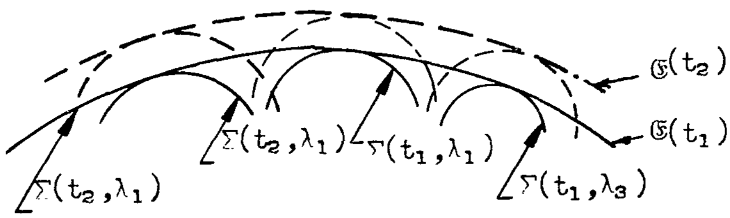
Corollary: The sound rays are the extremals of /IV,14

$$T(A, B; \Gamma) = \int_{A\Gamma B} \frac{d\sigma}{V_0 \cdot \frac{dM}{d\sigma} + \sqrt{c_0^2 - V_0^2 + (V_0 \cdot \frac{dM}{d\sigma})^2}} \tag{41}$$

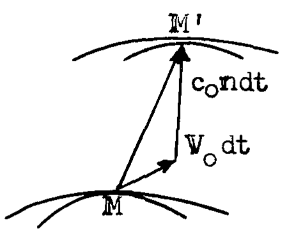
in the case in which $V_0 < c_0$.

R.C.W. A.C.W.

Centered waves: These are the various positions of a wave surface which, during its propagation, is reduced to a point. A differentiation must be made



between radiant centered waves that deviate from the center as time passes and antiradiant centered waves that approach the center.

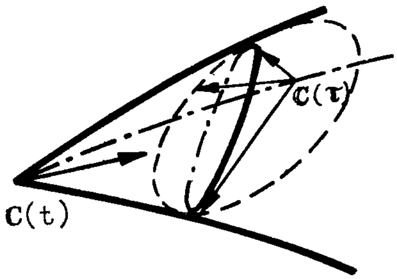


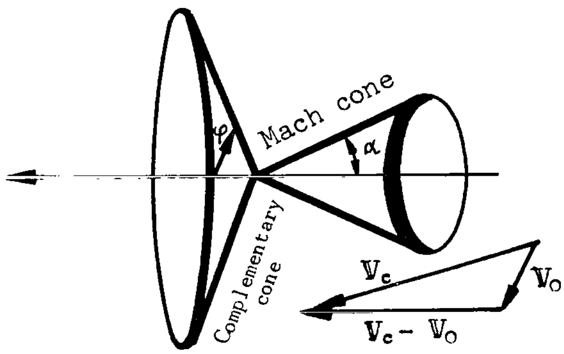
Theorem 3: Let us consider wave surfaces $\Sigma(t, \lambda)$ that depend on the parameter λ. Let $\mathfrak{S}(t)$ be the envelope of $\Sigma(t, \lambda)$ at varying λ and fixed t, with $\mathfrak{S}(t)$ being a wave surface.

The proof results directly from the propagation law as indicated in the

accompanying diagram.

Generalized Mach waves. This is the envelope of a family of centered waves R for the case that the center describes a curvilinear and arbitrary trajectory at supersonic velocity. The accompanying diagram shows the position of the Mach surface at the instant t, representing a conoid having its vertex in $\mathbf{C}(t)$ which is the position of the center at the same instant; the cone, /IV,15





$$\sin \alpha = \frac{c_0}{|\mathbf{V}_0 - \mathbf{V}_c|}$$

$$\cos \varphi = \frac{c_0}{|\mathbf{V}_0 - \mathbf{V}_c|}$$

tangent in $\mathbf{C}(t)$ to this conoid, is a cone of revolution whose axis coincides with the tangent to the trajectory if $\mathbf{V}_0$ is zero or with the rectilinear base of the vector $\mathbf{V}_c - \mathbf{V}_0$ where $\mathbf{V}_c$ is the velocity vector of the center; it has $\sin^{-1} \frac{c_0}{|\mathbf{V}_c - \mathbf{V}_0|}$ as half-aperture angle and is directed toward the anterior positions of the center. The accompanying diagram shows the Mach surface with a vertex $\mathbf{C}(t)$ and the position, at the instant t, of the centered wave with the center $\mathbf{C}(\tau)$; here, τ is the instant of emission of the centered wave. We also plotted the curve of contact with the envelope.

Theorem 4: The locus of the contact points of a centered wave with its envelope is a sound-ray conoid having its vertex at the center whose tangent

cone is complementary to the Mach cone.

Proof: The contact points are displaced along the rays in accordance with the proof of theorem 3; the tangent cone at the instant of emission obviously is complementary to the cone tangent to the envelope.

4.1.4 Analogy with Point Dynamics

Let $\psi(t, \mathbf{x}) = 0$ be the equation of a wave. The fundamental law of propagation is represented by /IV,16

$$\frac{\partial \psi}{\partial t} + (\mathbf{V} + c\,\mathbf{n}) \cdot \nabla \psi = 0 \quad , \tag{42}$$

on agreeing, for the duration of $u\xi$, to replace c_0 and $\mathbf{V}_0$ by c and $\mathbf{V}$. Since $\nabla\psi = \pm\,\mathbf{n}\,|\nabla\psi|$, we obtain

$$\left(\frac{\partial \psi}{\partial t} + \mathbf{V} \cdot \nabla \psi\right)^2 - c^2 |\nabla \psi|^2 = 0 \,. \tag{43}$$

As already done in Chapter I, let us introduce the coordinates

$$x^\alpha = (x^0, x^1, x^2, x^3) = (x^0, \mathbf{x}) = (t, \mathbf{x}) , \tag{44}$$

and pose

$$\frac{\partial \psi}{\partial x^\alpha} = p_\alpha = (p_0, p_1, p_2, p_3) = (p_0, \mathbf{p}) \quad , \tag{45}$$

in such a manner that eq.(43) can be written as follows:

$$0 = H(p_\alpha; x^\alpha) = g^{\alpha\beta}(x^\gamma)\, p_\alpha p_\beta = (p_0 + \mathbf{V} \cdot \mathbf{p})^2 - c^2 |\mathbf{p}|^2 . \tag{46}$$

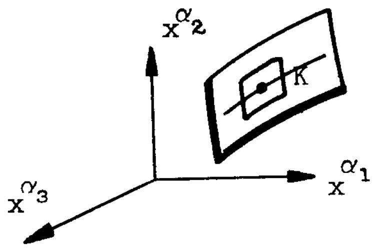
Let $\psi(x^a)$ be a solution function of eq.(43) and, on the surface ψ = const, let us consider a curve K (we are reasoning in the four-dimensional space of variables x^a) and let us specify this function by imposing a condition as to its direction in each of its points. Let u^a be a four-component vector which defines this direction; we obviously have $u^a p_a = 0$ but this is not sufficient for specifying the direction in question. To obtain this definition, let us imagine infinitesimal perturbations δp_a on all p_a, such that $H(p_a + \delta p_a) = 0\,((\delta p_a)^2)$ and let us impose the condition $u^a \delta p_a = 0$ for all perturbations in question. Thus, the quantities u^a are defined by

$$p_\alpha u^\alpha = 0 ,$$
$$\delta p_\alpha u^\alpha = 0 , \qquad \text{for} \qquad \frac{\partial H}{\partial p_\alpha} \delta p_\alpha = 0 , \tag{47}$$

It is then obvious that a unique solution /IV.17

$$u^\alpha = \frac{1}{2} \frac{\partial H}{\partial p_\alpha} \tag{48}$$

exists, to within a multiplicative factor. Let us recall that ψ is defined for any value of x^α and that this is true also for all p_α, from which it follows



that $H(p_\beta(x^\alpha), x^\alpha) \equiv 0$ is an identity and remains an identity by total differentiation in x^α, i.e.,

$$\sum_\beta \frac{\partial H}{\partial p_\beta} \frac{\partial p_\beta}{\partial x^\alpha} + \frac{\partial H}{\partial x^\alpha} = 0 . \tag{49}$$

However,

$$\frac{\partial p_\beta}{\partial x^\alpha} = \frac{\partial^2 \psi}{\partial x^\alpha \partial x^\beta} = \frac{\partial p_\alpha}{\partial x^\beta} , \tag{50}$$

so that we find

$$\sum_\beta \frac{\partial p_\alpha}{\partial x^\beta} u^\beta + \frac{1}{2} \frac{\partial H}{\partial x^\alpha} = 0 . \qquad (\alpha = 0, 1, 2, 3) \tag{51}$$

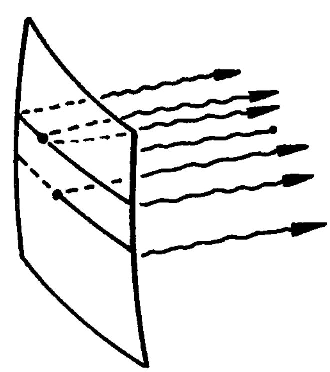
We now have the possibility of forming a differential system, defining the variation in x^α, p_α along the curve K, independently of the function ψ. For this, let s be a suitable parameter of K, so that

$$\left\{\begin{aligned} \frac{dx^\alpha}{ds} &= \frac{1}{2}\frac{\partial H}{\partial p_\alpha}, \\ \frac{dp_\alpha}{ds} &= -\frac{1}{2}\frac{\partial H}{\partial x^\alpha}, \end{aligned}\right. \tag{52}$$

which can be written explicitly as follows:

$$\left\{\begin{aligned} \frac{dt}{ds} &= p_0 + \mathbb{V}\cdot\mathbb{p}, \\ \frac{d\mathbb{x}}{ds} &= (p_0 + \mathbb{V}\cdot\mathbb{p})\,\mathbb{V} - c^2\,\mathbb{p}, \\ \frac{dp_0}{ds} &= -(p_0 + \mathbb{V}\cdot\mathbb{p})\frac{\partial\mathbb{V}}{\partial t}\cdot\mathbb{p} + |\mathbb{p}|^2 c\frac{\partial c}{\partial t}, \\ \frac{d\mathbb{p}}{ds} &= -(p_0 + \mathbb{V}\cdot\mathbb{p})(\nabla\mathbb{V})\cdot\mathbb{p} + |\mathbb{p}|^2 c\,\nabla c\,. \end{aligned}\right. \tag{53}$$

This differential system can be integrated independently of any considera- /IV,18
tion referring to eq.(43); this will then yield an eight-parameter network of



curves and planes tangent to these curves, having direction parameters p_α. It is immediately obvious that, along any integral curve, $H(p_\alpha, x^\alpha) = \text{const.}$ By eliminating the constant, we will still have a network of curves and planes tangent to seven parameters. In this seven-parameter network, let us isolate a subnet of three parameters, using formulas of the type

$$\begin{aligned} x^\alpha &= x^\alpha(a, s) \\ p_\alpha &= p_\alpha(a, s) \end{aligned} \qquad a = (a_1, a_2, a_3) \tag{54}$$

for example, a_1, a_2, a_3, yielding a system of curvilinear coordinates on a (space-time) surface which intersects the K of the subnet, so that one can select s = 0 at the point of intersection on each curve. Let us refer to the accompanying diagram for a geometric image. Let us use an arbitrary function $f(a)$ and let us suppose eq.(54) to be selected in such a manner that

$$\frac{\partial f}{\partial a_i} = p_\alpha(a_\kappa, 0)\,\frac{\partial x^\alpha(a_\kappa, 0)}{\partial a_i}, \qquad i = 1, 2, 3 \tag{55}$$

which is always possible since this reduces merely to selecting the initial values for eq.(53). We leave to the reader the task of proving that the function $\psi(t, x)$ obtained by elimination of a and s between

$$\left\{\begin{aligned} x^\alpha &= x^\alpha(a, s), \\ p_\alpha &= p_\alpha(a, s), \\ \psi &= f(a), \end{aligned}\right. \tag{56}$$

is such that

$$d\psi - p_\alpha\, dx^\alpha \equiv 0, \tag{57}$$

which verifies eq.(43). It is obvious that the condition /IV,19

$$H\left(p_\alpha(a, 0);\ x^\alpha(a, 0)\right) = 0 \tag{58}$$

is assumed to be satisfied. Conversely, any solution function $\psi(t, x)$ of eq.(43) can be obtained by this same process.

Problem 1: Use the integral curves of eq.(53) for solving

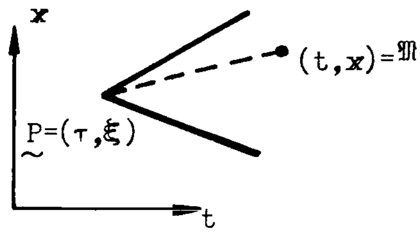
$$H\left(\frac{\partial \psi}{\partial x^\alpha}, x^\alpha\right) = \text{Const.}, \tag{59}$$

$$H\left(\frac{\partial \psi}{\partial x^\alpha}, x^\alpha\right) + \frac{\partial \psi}{\partial \sigma} = 0. \tag{60}$$

Theorem 5: If two integral surfaces of eq.(43) are tangent in a point, they will also be tangent over the entire length of the curve K which passes through this point and whose tangent plane, adjoint to this point, is the same as the tangent plane common to both surfaces.

The proof is evident (note that a multiplication of all p_α by const reduces to dividing s by the same const!). Compare this with theorem 4.

Characteristic conoid of vertex $\underset{\sim}{P} = (\tau, \xi)$. This is the name given to the surface generated by the curves K, issuing from the instant point $\underset{\sim}{P} = (\tau, \xi)$ and verifying the condition $H(p_\alpha, x^\alpha) = 0$.



Geodesic distance between $\underset{\sim}{P} = (\tau, \xi)$ and $\mathfrak{M} = (t, x)$. This is the name given to the function

$$\left\{\Gamma(\tau, \xi; t, x)\right\}^{\frac{1}{2}} = (s_{\mathfrak{M}} - s_{P}) H_0, \tag{61}$$

where H_0 denotes the constant value of $H(p_\alpha, x^\alpha)$ along the integral curve of eq.(53) which joins $\underset{\sim}{P}$ to $\mathfrak{M}$.

Let us explain the designation "geodesic distance". Let us consider /IV,20

$$\begin{cases} H(p_0, \mathbf{p}) \equiv (p_0 + \mathbb{V}\cdot\mathbf{p})^2 - c^2|\mathbf{p}|^2, \\ u^0 = p_0 + \mathbb{V}\cdot\mathbf{p}, \\ \mathbf{u} = (p_0 + \mathbb{V}\cdot\mathbf{p})\mathbb{V} - c^2\mathbf{p}, \end{cases} \tag{62}$$

by solving for p_0 and $\mathbf{p}$ and by substituting in H, which yields

$$H(p_0, \mathbf{p}) \doteq T(u^0, \mathbf{u}) \equiv (u^0)^2 - \frac{1}{c^2}|\mathbf{u} - u^0\mathbb{V}|^2 \tag{63}$$

from which it is obvious that

$$\Gamma(P; \mathfrak{M}) = \left\{\int_P^{\mathfrak{M}} T^{1/2}(dt, dx)\right\}^2 = \left\{\int_P^{\mathfrak{M}} T^{\frac{1}{2}}\left(\frac{dt}{ds}, \frac{dx}{ds}; t, x\right) ds\right\} \tag{64}$$

where the integral is always evaluated along the curve K joining P to $\mathfrak{M}$. This definition holds only for the curves K with $H > 0$. For the curves K with $H < 0$, we must write

$$\Gamma(\mathcal{P};\mathcal{M}) = -\left\{\int_{\mathcal{P}}^{\mathcal{M}}\left(-T(dt,dx)\right)^{\frac{1}{2}}\right\}^{2}. \tag{65}$$

Theorem 6: The integral curves of eq.(53) are also those that render the following integral stationary:

$$I = \int_{\mathcal{P}}^{\mathcal{M}} \left|T(dt,dx)\right|^{1/2}, \tag{66}$$

and are known as "geodesics".

Along the curve joining $\mathcal{P}$ to $\mathfrak{M}$, let us take a parameter s and let us pose $\frac{dx^a}{ds} = x^a$, which will yield, for the extremal curve

$$\frac{d}{ds}\left(\frac{\partial |T|^{\frac{1}{2}}}{\partial \dot{x}^{\alpha}}\right) - \frac{\partial |T|^{\frac{1}{2}}}{\partial x^{\alpha}} = 0 \quad ; \tag{67}$$

Now, the function $|T|^{\frac{1}{2}}$ is homogeneous and of the first degree relative to x^a, so that

$$\dot{x}^{\alpha}\frac{\partial |T|^{\frac{1}{2}}}{\partial \dot{x}^{\alpha}} = 1 \;, \tag{68}$$

from which we derive

/IV,21

$$\frac{d\dot{x}^{\alpha}}{ds}\frac{\partial |T|^{\frac{1}{2}}}{\partial \dot{x}^{\alpha}} + \dot{x}^{\alpha}\frac{d}{ds}\frac{\partial |T|^{1/2}}{\partial \dot{x}^{\alpha}} = 0 \;, \tag{69}$$

which shows that, for any integral curve of eq.(67), we have $|T|^{\frac{1}{2}}$ = const. From this it follows, except possibly for the curves that cancel T, that the integral curves of eq.(67) are also integral curves of

$$\frac{d}{ds}\left(\frac{\partial T}{\partial \dot{x}^{\alpha}}\right) - \frac{\partial T}{\partial x^{\alpha}} = 0 \;. \tag{70}$$

Thus, let us use an integral curve of eq.(70) and pose $p_a = \frac{1}{2}\frac{\partial T}{\partial \dot{x}^a}$, yielding $\dot{x}^a = \frac{1}{2}\frac{\partial H}{\partial p_a}$ such that eqs.(52) and thus also eq.(53) are verified, as results from

$$\frac{\partial H}{\partial x^\alpha} + \frac{\partial T}{\partial x^\alpha} = 0 \quad . \tag{71}$$

For proving this relation, the reader should remember that, in $\frac{\partial H}{\partial x^\alpha}$, the quantities p are maintained constant whereas, in $\frac{\partial T}{\partial x^\alpha}$, the quantities u in $\dot{x}$ are maintained constant. The case of T = 0 is also covered by theorem 6, but it is obvious that the extremals of eq.(66) are then poorly defined. Thus, if $\mathfrak{M}$ is located on the characteristic conoid of vertex $\underset{\sim}{P}$, then any curve traced on this conoid and joining $\underset{\sim}{P}$ to $\mathfrak{M}$ will be an extremal of eq.(66).

Specific case. If c and $\mathbb{V}$ are constant, the integral curves of eq.(53) or the so-called geodesics will be rectilinear, yielding

$$\Gamma = (t-\tau)^2 - \frac{1}{c^2}\left| x - \xi - \mathbb{V}(t-\tau)\right|^2 . \tag{72}$$

Let us now, in the general case, look for the partial derivatives of the function $\Gamma(\xi^\alpha; x^\alpha)$; for this, let us use a parameter σ such that $\sigma = 0$ in ξ^α and $\sigma = 1$ in x^α, thus yielding

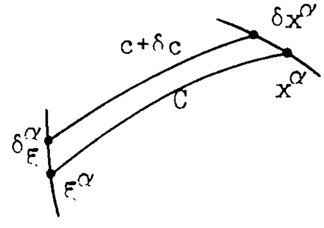
/IV,22

$$\Gamma(\xi^\alpha; x^\alpha) = \int_{\mathcal{P}}^{\mathfrak{M}} T\left(\frac{dz^\alpha}{d\sigma}; z^\alpha\right) d\sigma \, , \tag{73}$$

and, by variation of the extremities and of the connecting curve,

$$\frac{\partial \Gamma}{\partial x^\alpha}\delta x^\alpha + \frac{\partial \Gamma}{\partial \xi^\alpha}\delta \xi^\alpha = \left(\frac{\partial T}{\partial \dot{z}^\alpha}\delta z^\alpha\right)_{\sigma=0}^{\sigma=1} \tag{74}$$

because of eq.(71).



Theorem 7: The partial derivatives of the function $\Gamma(\tau, \xi; t, x)$ are given by

$$
\begin{cases}
\dfrac{\partial \Gamma}{\partial t} = \dfrac{\partial T\left(\frac{dt}{ds}, \frac{d\mathbf{x}}{ds}; t, \mathbf{x}\right)}{\partial \frac{dt}{ds}}, \\[2ex]
\dfrac{\partial \Gamma}{\partial \mathbf{x}} = \dfrac{\partial T\left(\frac{dt}{ds}, \frac{d\mathbf{x}}{ds}; t, \mathbf{x}\right)}{\partial \frac{d\mathbf{x}}{ds}}, \\[2ex]
\dfrac{\partial \Gamma}{\partial \tau} = -\dfrac{\partial T\left(\frac{d\tau}{ds}, \frac{d\xi}{ds}; \tau, \xi\right)}{\partial \frac{d\tau}{ds}}, \\[2ex]
\dfrac{\partial \Gamma}{\partial \xi} = -\dfrac{\partial T\left(\frac{d\tau}{ds}, \frac{d\xi}{ds}; \tau, \xi\right)}{\partial \frac{d\xi}{ds}},
\end{cases} \tag{75}
$$

in a parametrization of the geodesic joining (τ, ξ) to $(t, \mathbf{x})$, for which $s = 0$ in (τ, ξ) and $s = 1$ in $(t, \mathbf{x})$. In addition, we have

$$
\begin{cases}
\left(\dfrac{\partial \Gamma}{\partial t} + \mathbf{V}\cdot\dfrac{\partial \Gamma}{\partial \mathbf{x}}\right)^2 - c^2\left|\dfrac{\partial \Gamma}{\partial \mathbf{x}}\right|^2 = 4\Gamma, \\[2ex]
\left(\dfrac{\partial \Gamma}{\partial \tau} + \mathbf{V}\dfrac{\partial \Gamma}{\partial \xi}\right)^2 - c^2\left|\dfrac{\partial \Gamma}{\partial \xi}\right|^2 = 4\Gamma.
\end{cases} \tag{76}
$$

Let us now return to the geodesics as integrals of the system (53) and let us consider those for which

$$
H \equiv (p_0 + \mathbf{V}\cdot\mathbf{p})^2 - c^2|\mathbf{p}|^2 = 0 \tag{77}
$$

posing

/IV,23

$$
\mathbf{p} = -|\mathbf{p}|\,\mathbf{n}, \qquad |\mathbf{n}| = 1. \tag{78}
$$

We obtain

$$
\begin{cases}
\dfrac{d\mathbf{x}}{dt} = \mathbf{V} + c\,\mathbf{n}, \\[2ex]
\dfrac{d\mathbf{n}}{dt} = -\nabla c - (\nabla \mathbf{V})\cdot\mathbf{n} + \mathbf{n}\,\dfrac{d|\mathbf{p}|}{dt},
\end{cases} \tag{79}
$$

if we assume $\frac{dt}{ds} > 0$.

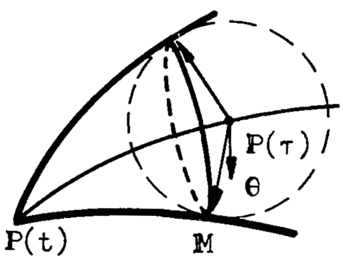
Theorem 8: The sound rays defined in Section 4.1.3 are spatial projections of the geodesics of zero length.

4.1.5 Special Cases

If $\mathbb{V}_0$ is zero and if c_0 is time-independent, we have ($d\sigma$ = element of arc)

$$\Gamma(\tau, \mathbb{P}; t, \mathbb{M}) = (t-\tau)^2 - \left(\int_{\mathbb{P}}^{\mathbb{M}} \frac{d\sigma}{c_0}\right)^2 = (t-\tau)^2 - \left(T(\mathbb{P}, \mathbb{M})\right)^2 \tag{80}$$

where the integral is evaluated along a sound ray which renders the integral in question extremal. It is obvious that $T(\mathbb{P}, \mathbb{M})$ is the transit time of a sound



wave of $\mathbb{P}$ in $\mathbb{M}$. Let us assume that $\mathbb{P}$ is displaced along a trajectory in accordance with the hour law $\mathbb{P}(\tau)$ and let us pose

$$\Gamma(\tau, \mathbb{P}(\tau); t, \mathbb{M}) = \tilde{\Gamma}(\tau; t, \mathbb{M}), \tag{81}$$

The Mach wave, if it exists at all, is given by

$$\begin{cases} \tilde{\Gamma}(\tau; t, \mathbb{M}) = 0 , \\ \dfrac{\partial \tilde{\Gamma}(\tau; t, \mathbb{M})}{\partial \tau} = 0 . \end{cases} \tag{82}$$

Let us denote by Θ the unit vector tangent in $\mathbb{P}$ to the sound ray joining $\mathbb{P}$ and $\mathbb{M}$, so that

$$\frac{d\, T(\mathbb{P}(\tau), \mathbb{M})}{d\tau} = - \frac{\Theta \cdot \mathbb{V}_{\mathbb{P}}}{c_0(\mathbb{P})} \qquad \mathbb{V}_{\mathbb{P}} = \frac{d\mathbb{P}}{d\tau} \tag{83}$$

and, consequently, /IV,24

$$\frac{1}{2}\frac{\partial\tilde{\Gamma}}{\partial\tau} = \tau - t + \frac{\Theta \cdot \mathbb{V}_P}{C_0(P)}\int_P^M \frac{d\sigma}{C_0} . \tag{84}$$

Thus, eq.(82) implies

$$\frac{\Theta \cdot \mathbb{V}_P}{C_0(P)} = 1 , \tag{85}$$

in conformity with theorem 4.

If we have, simultaneously,

$$\tilde{\Gamma} = \frac{\partial\tilde{\Gamma}}{\partial\tau} = \frac{\partial^2\tilde{\Gamma}}{\partial\tau^2} = 0 , \tag{86}$$

then the Mach conoid exhibits an inflection at the corresponding point. In general, this takes place over an entire line at a predetermined instant t. Let us study the specific case in which a_0 is constant, and let us pose

$$\frac{dP}{d\tau} = \mathbb{V}_P , \qquad \frac{d^2P}{d\tau^2} = \mho_P . \tag{86a}$$

We then have

$$\left\{\begin{aligned} \tilde{\Gamma} &= (t-\tau)^2 - \frac{|PM|^2}{C_0^2} , \\ \frac{1}{2}\frac{\partial\tilde{\Gamma}}{\partial\tau} &= \tau - t + \frac{PM \cdot \mathbb{V}_P}{C_0^2} , \\ \frac{1}{2}\frac{\partial^2\tilde{\Gamma}}{\partial\tau^2} &= 1 - \frac{|\mathbb{V}_P|^2}{C_0^2} - \frac{\mho_P \cdot PM}{C_0^2} , \end{aligned}\right. \tag{87}$$

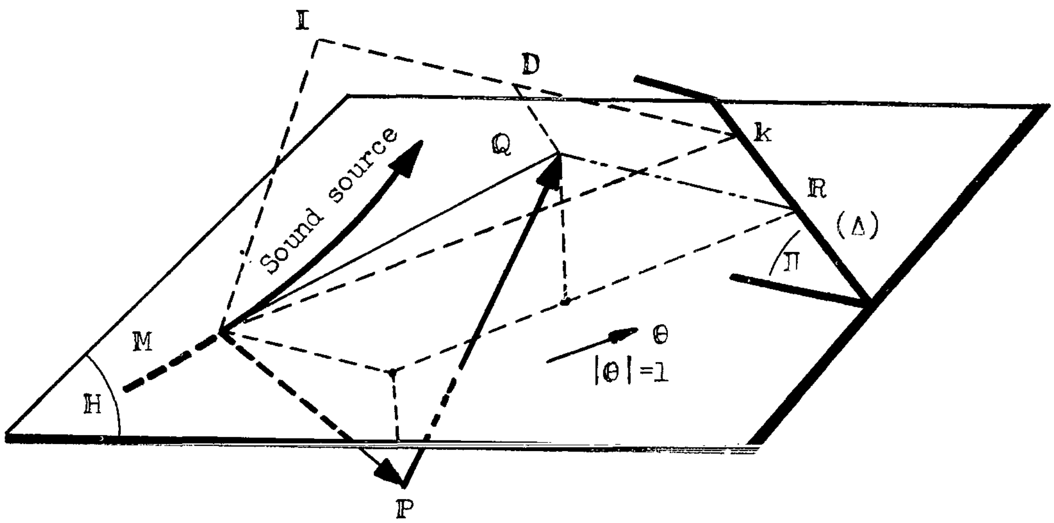
meaning that the presence of an inflection of the Mach wave is connected with

the acceleration of the center in a homogeneous atmosphere.

Time-independent stratified atmosphere. This is the designation given to an atmosphere in which V_0 and c_0 are constants in planes perpendicular to a fixed vector. We will devote the end of this particular Section to a /IV,25
study of some properties of sound rays in such an atmosphere. Let k denote the unit vector normal to the stratification planes; in accordance with theorem 1, the vector $\frac{d\mathbf{n}}{dt}$ ($\mathbf{n}$ is always the unit vector normal to the wave in the direction of propagation) is orthogonal to n and to $\mathbf{k} \wedge \mathbf{n}$, so that we can write $\frac{d\mathbf{n}}{d\lambda} = \lambda\, \mathbf{n} \wedge (\mathbf{k} \wedge \mathbf{n})$ where λ is a scalar. From this, we derive

$$\frac{d\, \mathbf{k}\wedge\mathbf{n}}{dt} = \lambda\, \mathbf{k}\wedge[\mathbf{n}\wedge(\mathbf{k}\wedge\mathbf{n})] = \lambda(\mathbf{k}\cdot\mathbf{n})\ \mathbf{k}\wedge\mathbf{n}, \tag{88}$$

which shows that the vector $\mathbf{k} \wedge \mathbf{n}$ retains a fixed direction. In the absence of wind, the sound rays are plane curves traced in planes orthogonal to the stratification plane. Thus, we have obtained a first integral of the differential system, defining the sound rays. Still another integral exists which can be



formulated as follows:

Theorem 9: At a moving point $\mathbf{M}$ of the sound ray, let us trace this plane (H) parallel to the plane of stratification and let us consider the contour $\mathbf{M}\,\mathbf{P}\,\mathbf{Q}$ with $\mathbf{M}\,\mathbf{P} = \mathbf{V}_0$, $\mathbf{P}\,\mathbf{Q} = c_0\mathbf{n}$ and let $\mathbf{Q}$ be the plane normal to $\mathbf{n}$ which is supposed to intersect (H) along a straight line (Δ). The distance δ from the point $\mathbf{M}$ to the straight line (Δ) remains constant when $\mathbf{M}$ traverses this sound ray.

Let us use the notations of theorem 1 and those of the accompanying diagram, denoting by $\mathbf{U}_n$ the projection of $\mathbf{V}_0$ onto the plane (Π), yielding (making use of the stratification!)

/IV,26

$$\frac{dw}{dt} = \mathbb{K}\,\mathbb{J}\,.\,\nabla_\tau w_n + \mathbb{V}_0 . \frac{d\mathbf{n}}{dt} . \tag{89}$$

If, as suggested by theorem 9, the quantities δ and Θ [unit vector of the projection of $\mathbf{n}$ onto (H)] are constant, we will also have

$$\frac{dw}{dt} = \delta\Theta . \frac{d\mathbf{n}}{dt} , \tag{90}$$

from which, by comparison of eqs.(89) and (90), we obtain

$$\mathbb{K}\,\mathbb{J} . \left(\nabla_\tau w_n + \frac{d\mathbf{n}}{dt} \right) = 0 , \tag{91}$$

in conformity with theorem 1.

Corollary: In a stratified atmosphere without wind, we will have

$$\frac{C_0}{\cos\theta} = \mathrm{const} , \tag{92}$$

along a sound ray, where θ is the angle made by the tangent to the sound ray with the orthogonal direction to the plane of stratification.

4.1.6 Acoustic Field Concentrated on a Surface

Let us return to the general equations of acoustics (2) and let us assume very regular ρ_0, p_0, $\mathbf{V}_0$, S_0 so that p_1, ρ_1, S_1, $\mathbf{V}_1$ can be distributions. Thus, let us assume

$$\begin{cases} \mathbb{V}_1 = \mathbb{A}\,\mathfrak{V}(F) + \cdots , \\ \rho_1 = B\,\mathfrak{R}(F) + \cdots , \\ S_1 = C\,\Sigma(F) + \cdots , \end{cases} \tag{93}$$

where $\mathfrak{B}$, $\mathfrak{R}$, Σ are derivatives of $\delta(F)$ with respect to F, where these points have the same form but have derivatives of a less high order and possibly also have functions ($\mathbf{A}$ being a vector and B and C being scalars) which are regular functions of t and $\mathbf{x}$ that do not vanish on $F = 0$. After substitution in the equations of motion and making use of /IV,27

$$\frac{\partial \mathcal{T}(F)}{\partial x^\alpha} = \frac{\partial F}{\partial x^\alpha}\frac{d\mathcal{T}(F)}{dF}, \tag{94}$$

we obtain

$$\begin{cases} \rho_0\left(\frac{\partial F}{\partial t} + \mathbb{V}_0\cdot\nabla F\right)\mathbb{A}\,\mathfrak{V}'(F) + c_0^2\,\nabla F\;\mathbb{B}\,\mathfrak{R}'(F) + g_{s_0}\nabla F\;C\,\Sigma'(F) + \dots = 0, \\ \rho_0\,\mathbb{A}\cdot\nabla F\;\mathfrak{V}'(F) + \left(\frac{\partial F}{\partial t} + \mathbb{V}_0\cdot\nabla F\right)\mathbb{B}\;\mathfrak{R}'(F) + \dots = 0, \\ \left(\frac{\partial F}{\partial t} + \mathbb{V}_0\cdot\nabla F\right)C\;\Sigma'(F) + \dots = 0, \end{cases} \tag{95}$$

where the points do not contain derivatives of $\mathfrak{V}$, $\mathfrak{R}$, or Σ. More accurately, let

$$\mathfrak{V} = \frac{d^a\delta(F)}{dF^a}, \quad \mathfrak{R} = \frac{d^b\delta(F)}{dF^b}, \quad \Sigma = \frac{d^c\delta(F)}{dF^c}. \tag{96}$$

If $c \geq \sup(a, b)$, then the third equation in the system (95) shows that we necessarily have*

$$\frac{\partial F}{\partial t} + \mathbb{V}_0\cdot\nabla F = 0, \tag{97}$$

where the surface $F = 0$ is an entropy wave. The first equation of the system (95) consequently shows that we should have $c = b$ and

$$c_0^2\,\mathbb{B} + g_{s_0}\,C = 0, \qquad \text{on } F = 0. \tag{98}$$

Finally, if $c = a > b$, the second equation of the system (95) requires that

$$\mathbb{A}\cdot\nabla F = 0. \tag{99}$$

However, if $c = b > a$, we cannot conclude. Let us now consider the case in which $c < \sup(a, b)$, so that we have

$$\begin{cases} \rho_0\left(\frac{\partial F}{\partial t} + \mathbb{V}_0\cdot\nabla F\right)\mathbb{A}\,\mathfrak{V}'(F) + c_0^2\,\nabla F\;\mathbb{B}\,\mathfrak{R}'(F) + \dots = 0 \\ \rho_0\,\mathbb{A}\cdot\nabla F\;\mathfrak{V}'(F) + \left(\frac{\partial F}{\partial t} + \mathbb{V}_0\cdot\nabla F\right)\mathbb{B}\;\mathfrak{R}'(F) + \dots = 0 \end{cases} \tag{100}$$

---

* If we have $\alpha_0\,\delta^{(k)}(F) + \alpha_1\,\delta^{(k-1)}(F) + \dots = 0$, we also must have $\alpha_0 = 0$ on $F = 0$.

with the neglected terms being of lower order. If $a < b$, we necessarily will have $\nabla F = 0$, $\frac{\partial F}{\partial t} = 0$ on $F = 0$, which is excluded, so that $a \geq b$. If /IV,28 $a = b$, it is necessary that

$$\left(\frac{\partial F}{\partial t} + \mathbb{V}_0 \cdot \nabla F\right)^2 - c_0^2 |\nabla F|^2 = 0, \tag{101}$$

where the surface $F = 0$ is an acoustic wave, yielding

$$\begin{array}{ll} \nabla F = \lambda\, \mathbb{n} & \frac{\partial F}{\partial t} + \mathbb{V}_0 \cdot \nabla F = c_0 \lambda \\ \mathbb{A} = c_0 \phi\, \mathbb{n} & \mathbb{B} = -\rho_0 \Phi \end{array} \tag{102}$$

on $F = 0$. If $a > b$, it follows that

$$\frac{\partial F}{\partial t} + \mathbb{V}_0 \cdot \nabla F = 0 \quad , \qquad \mathbb{A} \cdot \nabla F = 0 \tag{103}$$

again involving an entropy wave.

The above reasoning demonstrates that we are covering all possible cases by posing the following:

Acoustic wave:

$$\left\{ \begin{array}{l} \mathbb{V}_1 = (c_0 \phi\, \mathbb{n} + F \mathbb{A}^0)\, \mathcal{T}_0(F) + \mathbb{A}^1 \mathcal{T}_1(F) + \mathbb{A}^2 \mathcal{T}_2(F) + \cdots, \\ \rho_1 = (-\rho_0 \phi + F \mathbb{B}^0)\, \mathcal{T}_0(F) + \mathbb{B}^1 \mathcal{T}_1(F) + \mathbb{B}^2 \mathcal{T}_2(F) + \cdots, \\ S_1 = C^1 \mathcal{T}_1(F) + C^2 \mathcal{T}_2(F) + \cdots, \end{array} \right. \tag{104}$$

with the orders of the distributions $\mathcal{T}_i(F)$ decreasing on increasing i. The unit vector $\mathbb{n}$ is normal to $F = 0$ and selected in such a manner that

$$\frac{\partial F}{\partial t} + \mathbb{V}_0 \cdot \nabla F = c_0\, \mathbb{n} \cdot \nabla F. \tag{105}$$

Entropy wave:

$$\left\{ \mathbb{V}_1 = \mathbb{A}_T\, \mathcal{T}_0(F) + \mathbb{A}^1 \mathcal{T}_1(F) + \mathbb{A}^2 \mathcal{T}_2(F) + \cdots \right. \tag{106}$$

$$\begin{cases} \rho_1 = \left(g_{s_0}\phi + F\overset{0}{B}\right)\mathfrak{S}_0(F) + \overset{1}{B}\,\mathfrak{S}_1(F) + \overset{2}{B}\,\mathfrak{S}_2(F) + \cdots \\ S_1 = \left(-c_0^2\phi + F\overset{0}{C}\right)\mathfrak{S}_0(F) + \overset{1}{C}\,\mathfrak{S}_1(F) + \overset{2}{C}\,\mathfrak{S}_2(F) + \cdots \end{cases}$$

with

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$$\frac{\partial F}{\partial t} + \mathbb{V}_0 \cdot \nabla F = 0 \quad , \quad \mathbb{A}_T \cdot \nabla F = 0 \quad , \qquad \text{on} \quad F = 0 \quad . \tag{107}$$

The scalar or vectorial functions ϕ, A_T, B^0, C^0, A^1, ... are subject to certain differential relations which we will investigate further. For this purpose, it is convenient to modify eqs.(104) and (105). Let us note that we have

$$\begin{cases} F\,\delta(F) = 0 \;, \\ F\,\delta'(F) + \delta(F) = 0 \;, \\ F\,\delta''(F) + 2\delta'(F) = 0 \;, \\ \cdots \\ F\,\delta^{(K)}(F) + K\,\delta^{(K-1)}(F) = 0 \;, \end{cases} \tag{108}$$

which permits writing

$$\begin{cases} \mathfrak{S}_1(F) = -\frac{1}{K}\,F\,\mathfrak{S}_0(F) \\ \mathfrak{S}_2(F) = -\frac{1}{K(K-1)}\,F^2\,\mathfrak{S}_0(F) \end{cases} \tag{109}$$

if, for example, $\mathfrak{S}_0 = \dfrac{d^k\delta(F)}{dF^k}$.

Thus, we have the following formulas:

Acoustic wave:

$$\begin{cases} \mathbb{V}_1 = \left(c_0\phi\, n + \sum\limits_{K=0}^{K} F^{K+1}\mathbb{A}^K\right)\dfrac{d^K\delta(F)}{dF^K} + \mathbb{A}'\,1(F) + \mathbb{A}'' , \\ \rho_1 = \left(-\rho_0\phi + \sum\limits_{K=0}^{K} F^{K+1}B^K\right)\dfrac{d^K\delta(F)}{dF^K} + B'\,1(F) + B'' , \\ S_1 = \left(\sum\limits_{K=0}^{K} F^{K+1}C^K\right)\dfrac{d^K\delta(F)}{dF^K} + C'\,1(F) + C'' \end{cases} \tag{110}$$

with

$$\mathbb{1}(F) = \begin{cases} 1 & \text{if } F > 0 \\ 0 & \text{if } F < 0 \end{cases} \tag{111}$$

where $\mathbb{A}'$, ... are continuous in F = 0 while $\mathbb{A}^k$, ... do not depend on F.

Entropy wave: /IV,30

$$\begin{cases} \mathbb{V}_1 = \left(\mathbb{A}_T + \sum_{k=0}^{K} F^{k+1}\mathbb{A}^k\right)\dfrac{d^K\delta(F)}{dF^K} + \mathbb{A}'\,\mathbb{1}(F) + \mathbb{A}'', \\ \rho_1 = \left(g_{s_0}\phi + \sum_{k=0}^{K} F^{k+1}\mathbb{B}^k\right)\dfrac{d^K\delta(F)}{dF^K} + \mathbb{B}'\,\mathbb{1}(F) + \mathbb{B}'', \\ S_1 = \left(-c_0^2\phi + \sum_{k=0}^{K} F^{k+1}C^k\right)\dfrac{d^K\delta(F)}{dF^K} + C'\,\mathbb{1}(F) + C''. \end{cases} \tag{112}$$

However, it would be inelegant to substitute eq.(110) or eq.(112) in the equations of motion since this would mask the basic role to be played by the acoustic rays for eq.(110) and by the entropy rays for eq.(112).

4.2 Acoustic Equations in Characteristic Coordinates

4.2.1 Definition of the Coordinates

Let us consider a family of surfaces $\Sigma(t)$ which are displaced in time and whose parametric representation is

$$t = x_0 \quad , \qquad \mathbb{P} = \mathbb{P}(x_0, \mathbf{a}) \ , \tag{1}$$

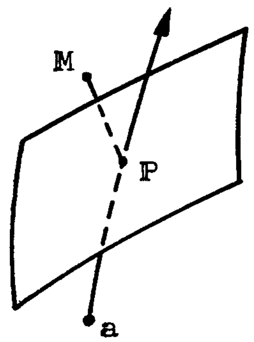
with, for example, $\mathbb{P}(o, \mathbf{a}) = \mathbf{a}$ where the point $\mathbf{a}$ is displaced over one surface of the family. Let us note that $\bar{n}(x_0, \mathbf{a}) = \mathbf{n}(x_0, \mathbb{P})$ is the unit vector normal in $\mathbb{P}$ to the surface which passes there at the instant $t = x_0$, selected in such a manner that

$$\mathbf{n} \cdot \frac{\partial \mathbb{P}}{\partial x_0} \geqslant 0 . \tag{2}$$

Let us represent a space-time point instant by

$$t = x_0 \qquad \mathbb{M} = \mathbb{P}(x_0, \mathbf{a}) + x_1\, \bar{n}(x_0, \mathbf{a}) \tag{3}$$

and let us select x_0, x_1, $\mathbb{P}$ as the system of coordinates. We will attempt to define as little as possible the coordinates used for specifying the position



of the point $\mathbb{P}$ on the surface $\Sigma(t)$. However, whenever we will need a coordinate system for performing some intermediary calculation, we will use orthogonal curvilinear coordinates (x_2, x_3) on $\Sigma(t)$ such that the coordinate lines will be the lines of curvature of this surface. Let us note, on $\Sigma(t)$ /IV,31

$$d\mathbb{P} = H_2 \mathbb{e}_2 \, dx_2 + H_3 \mathbb{e}_3 \, dx_3 \,, \tag{4}$$

where $\mathbb{e}_2$ and $\mathbb{e}_3$ are unitary, and let us denote by

$$\mathbb{K} = \frac{K_2}{H_2} \mathbb{e}_2 \mathbb{e}_2 + \frac{K_3}{H_3} \mathbb{e}_3 \mathbb{e}_3 \tag{5}$$

the curvature tensor of the surface $\Sigma(t)$ such that, for a displacement along the surface, we have

$$d\mathbb{n} = d\mathbb{P} . \mathbb{K} \,, \tag{6}$$

and, consequently,

$$d\mathbb{M} = \left(\frac{\partial \mathbb{P}}{\partial x_0} + x_1 \frac{\partial \mathbb{n}}{\partial x_0} \right) dx_0 + d\mathbb{P} . \left(\mathbb{I}_\pi + x_1 \mathbb{K} \right) + \mathbb{n} \, dx_1 \,, \tag{7}$$

by noting that

$$\mathbb{I}_\pi = \mathbb{e}_2 \mathbb{e}_2 + \mathbb{e}_3 \mathbb{e}_3 \tag{8}$$

is the unit tensor in the plane tangent to $\Sigma(t)$. It is convenient to make use also of the tensor

$$\mathbb{H} = \frac{H_2}{H_2 + x_1 K_2}\,\mathbb{e}_2\mathbb{e}_2 + \frac{H_3}{H_3 + x_1 K_3}\,\mathbb{e}_3\mathbb{e}_3 \tag{9}$$

and of the surface gradient vector

$$\mathbb{D} = \frac{\mathbb{e}_2}{H_2}\frac{\partial}{\partial x_2} + \frac{\mathbb{e}_3}{H_3}\frac{\partial}{\partial x_3}. \tag{10}$$

We leave to the reader the task of establishing, by passing over the intermediary of the coordinates x_2 and x_3, the following formulas:

$$\begin{cases} dx_0 = dt, \\ dx_1 = -\dfrac{\partial \mathbb{P}}{\partial x_0}\cdot \mathbb{n}\, dt + \mathbb{n}\cdot d\mathbb{M}, \\ d\mathbb{P} = -\mathbb{H}\cdot\left(\dfrac{\partial \mathbb{P}}{\partial x_0} + x_1 \dfrac{\partial \bar{\mathbb{n}}}{\partial x_0}\right) dt + \mathbb{H}\cdot d\mathbb{M}, \end{cases} \tag{11}$$

leading to the formulas for change in coordinates: /IV,32

$$\begin{cases} \dfrac{\partial}{\partial t} = \dfrac{\partial}{\partial x_0} - \mathbb{n}\cdot\dfrac{\partial \mathbb{P}}{\partial x_0}\dfrac{\partial}{\partial x_1} - \left(\dfrac{\partial \mathbb{P}}{\partial x_0} + x_1\dfrac{\partial \bar{\mathbb{n}}}{\partial x_0}\right)\cdot\mathbb{H}\cdot\mathbb{D}, \\ \nabla = \mathbb{n}\dfrac{\partial}{\partial x_1} + \mathbb{H}\cdot\mathbb{D}. \end{cases} \tag{12}$$

4.2.2 Case of an Acoustic Wave

Let us write now the equations of motion in the selected coordinate system for the case that the surface $\Sigma(t)$ is an acoustic wave surface. We then have

$$\begin{cases} \dfrac{\partial \mathbb{P}}{\partial x_0} = \mathbb{V}_0 + c_0\,\mathbb{n}, \\ \dfrac{\partial \bar{\mathbb{n}}}{\partial x_0} = -\mathbb{D}c_0 - (\mathbb{D}\mathbb{V}_0)\cdot\mathbb{n} \end{cases} \tag{13}$$

so that

$$\frac{\partial}{\partial t} + \mathbb{V}_0\cdot\nabla = \frac{\partial}{\partial x_0} - c_0\frac{\partial}{\partial x_1} + x_1\left(\mathbb{D}c_0 + (\mathbb{D}\mathbb{V}_0)\cdot\mathbb{n}\right)\cdot\mathbb{H}\cdot\mathbb{D}. \tag{14}$$

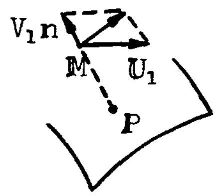
Before writing the acoustic equations, let us transform the equation of the momentum in

$$\rho_0\left(\frac{\partial}{\partial t} + \mathbb{V}_0\cdot\nabla\right)\mathbb{V}_1 + \nabla\left(c_0^2\rho_1 + g_{s_0} s_1\right) - \frac{\rho_1}{\rho_0}\left(c_0^2\nabla\rho_0 + g_{s_0}\nabla s_0\right) + \rho_0\mathbb{V}_1\cdot\nabla\mathbb{V}_0 = 0 \tag{15}$$

and let us pose

$$\mathbb{V}_1 = v_1\,\mathbb{n} + \mathbb{U}_1\,, \tag{16}$$

where the vector $\mathbb{U}_1$ is parallel to the plane tangent in $\mathbb{P}$ to $\Sigma(t)$. Using these notations, the acoustic equations can be written as follows, assuming - for



simplification - that $\frac{\partial S_0}{\partial t} + \mathbb{V}_0 \cdot \nabla S_0 = 0$:

/IV,33

$$\left\{\begin{aligned}
&\boxed{\left(c_0^2\frac{\partial\rho_1}{\partial x_1} + g_{s_0}\frac{\partial s_1}{\partial x_1} - \rho_0 c_0\frac{\partial v_1}{\partial x_1}\right)\mathbb{n} - \rho_0 c_0\frac{\partial\mathbb{U}_1}{\partial x_1}} + \rho_0\frac{\partial}{\partial x_0}\left(v_1\mathbb{n} + \mathbb{U}_1\right) + \\
&+ c_0^2\,\mathbb{H}\cdot\mathbb{D}\rho_1 + g_{s_0}\mathbb{H}\cdot\mathbb{D}s_1 + \rho_0 x_1\left(\mathbb{D}c_0 + (\mathbb{D}\mathbb{V}_0)\cdot\mathbb{n}\right)\cdot\mathbb{H}\cdot\mathbb{D}\left(v_1\mathbb{n} + \mathbb{U}_1\right) \\
&+\rho_1\nabla c_0^2 + s_1\nabla g_{s_0} - \frac{\rho_1}{\rho_0}\left(c_0^2\nabla\rho_0 + g_{s_0}\nabla s_0\right) + \rho_0 v_1\frac{\partial\mathbb{V}_0}{\partial x_1} + \rho_0\mathbb{U}_1\cdot\mathbb{H}\cdot\mathbb{D}\mathbb{V}_0 = 0\,, \\
&\boxed{\rho_0\frac{\partial v_1}{\partial x_1} - c_0\frac{\partial\rho_1}{\partial x_1}} + \frac{\partial\rho_1}{\partial x_0} + \rho_0(\mathbb{H}\cdot\mathbb{D})\cdot\mathbb{U}_1 + x_1\left(\mathbb{D}c_0 + (\mathbb{D}\mathbb{V}_0)\cdot\mathbb{n}\right)\cdot\mathbb{H}\cdot\mathbb{D}\rho_1 + \\
&\qquad + \rho_0 v_1(\mathbb{H}\cdot\mathbb{D})\cdot\mathbb{n} + v_1\frac{\partial\rho_0}{\partial x_1} + \mathbb{U}_1\cdot\mathbb{H}\cdot\mathbb{D}\rho_0 + \rho_1\nabla\cdot\mathbb{V}_0 = 0\,, \\
&\boxed{c_0\frac{\partial s_1}{\partial x_1}} - \frac{\partial s_1}{\partial x_0} - x_1\left(\mathbb{D}c_0 + (\mathbb{D}\mathbb{V}_0)\cdot\mathbb{n}\right)\cdot\mathbb{H}\cdot\mathbb{D}s_1 - v_1\frac{\partial s_0}{\partial x_1} - \mathbb{U}_1\cdot\mathbb{H}\cdot\mathbb{D}s_0 = 0.
\end{aligned}\right. \tag{17}$$

We leave to the reader the task of comparing the structure of the boxed terms in these formulas and in eqs.(13) of Section 4.1. Before going on, we must decompose the first equation of the system (17) which is a vectorial equation, by projecting it successively onto the vector $\mathbb{n}$ and onto the plane tangent to $\Sigma(t)$.

For this, let us note that we must be able to decompose $\frac{\partial \mathbf{n}}{\partial x_0}$, $\frac{\partial \mathbf{U}_1}{\partial x_0}$, $\mathbb{D}\,\mathbf{n}$, $\mathbb{D}\,\mathbf{U}_1$. So far as $\frac{\partial \mathbf{n}}{\partial x_0}$ is concerned, we only have the tangential component, while for $\frac{\partial \mathbf{U}_1}{\partial x_0}$ we must write the following decomposition:

$$\frac{\partial \mathbb{U}_1}{\partial x_0} = \frac{\partial \mathbb{U}_1}{\partial x_0} \cdot \mathbb{I}_T + \Omega_T \wedge \mathbb{U}_1 . \tag{18}$$

For determining Ω_T, let us use a moving orthonormal reference point $\mathbf{n}$, $\mathbf{E}_2$, $\mathbf{E}_3$ entrained by $\Sigma(t)$, yielding

$$\mathbb{U}_1 = U_{12}\,\mathbb{E}_2 + U_{13}\,\mathbb{E}_3 , \tag{19}$$

and

$$\frac{\partial \mathbb{U}_1}{\partial x_0} = \frac{\partial U_{12}}{\partial x_0}\,\mathbb{E}_2 + \frac{\partial U_{13}}{\partial x_0}\,\mathbb{E}_3 + U_{12}\,\frac{\partial \mathbb{E}_2}{\partial x_0} + U_{13}\,\frac{\partial \mathbb{E}_3}{\partial x_0} . \tag{20}$$

Then, let $\omega = \omega\,\mathbf{n} + \omega_T$ be the rotating vector of the reference point, so that we have

/IV,34

$$\frac{\partial \mathbf{n}}{\partial x_0} = \omega_T \wedge \mathbf{n} , \quad \frac{\partial \mathbb{E}_i}{\partial x_0} = \omega\, \mathbf{n} \wedge \mathbb{E}_i + \omega_T \wedge \mathbb{E}_i , \tag{21}$$

such that

$$\frac{\partial \mathbb{U}_1}{\partial x_0} = \frac{\partial U_{12}}{\partial x_0}\,\mathbb{E}_2 + \frac{\partial U_{13}}{\partial x_0}\,\mathbb{E}_3 + \omega\, \mathbf{n} \wedge \mathbb{U}_1 + \omega_T \wedge \mathbb{U}_1 , \tag{22}$$

and, consequently, $\Omega_T = \omega_T$, i.e.,

$$\Omega_T \wedge \mathbf{n} = -\mathbb{D}\,c_0 - (\mathbb{D}\,\mathbf{V}_0)\cdot \mathbf{n} , \tag{23}$$

from which it follows that

$$\Omega_T = -\mathbf{n} \wedge \mathbb{D}\,c_0 - \mathbf{n} \wedge (\mathbb{D}\,\mathbf{V}_0)\cdot \mathbf{n} , \tag{24}$$

and, finally,

$$\frac{\partial \mathbb{U}_1}{\partial x_0} = \frac{\partial \mathbb{U}_1}{\partial x_0} \cdot \mathbb{I}_T + \mathbf{n} \left\{ \mathbb{U}_1 \cdot \mathbb{D}\,c_0 + \mathbb{U}_1 \cdot (\mathbb{D}\,\mathbf{V}_0) \cdot \mathbf{n} \right\} . \tag{25}$$

Consequently, by definition of the curvature tensor, we have

$$\mathbb{D}\, \mathbb{n} = \mathbb{K}, \tag{26}$$

where the decomposition is trivial, so that it merely remains to decompose $\mathbf{D}\, \mathbf{U}_1$. We will proceed as in eq.(20)

$$\begin{aligned} \mathbb{D}\mathbb{U}_1 &= \cdots + U_{12}\, \mathbb{D}\mathbb{E}_2 + U_{13}\, \mathbb{D}\mathbb{E}_3 \\ &= (\mathbb{D}\mathbb{U}_1) \cdot \mathbb{I}_\tau + \left\{ U_{12} (\mathbb{D}\mathbb{E}_2) \cdot \mathbb{n} + U_{13} (\mathbb{D}\mathbb{E}_3) \cdot \mathbb{n} \right\} \mathbb{n} \end{aligned} \tag{27}$$

and note that, if $\mathbf{E}_2$ and $\mathbf{E}_3$ coincide with $\mathbf{e}_2$ and $\mathbf{e}_3$ introduced previously (this can be done since we operate with constant x_0!), we will have

$$(\mathbb{D}\mathbb{E}_2) \cdot \mathbb{n} = -\frac{K_2}{H_2}\, \mathbb{e}_2 \quad , \qquad (\mathbb{D}\mathbb{E}_3) \cdot \mathbb{n} = -\frac{K_3}{H_3}\, \mathbb{e}_3 , \tag{28}$$

and, consequently,

$$\mathbb{D}\mathbb{U}_1 = (\mathbb{D}\mathbb{U}_1) \cdot \mathbb{I}_\tau - \mathbb{K} \cdot \mathbb{U}_1\ \mathbb{n} \tag{29}$$

[this means $\mathbf{A}_\tau \cdot \mathbf{D}\, \mathbf{U}_1 = \mathbf{A}_\tau \cdot (\mathbf{D}\, \mathbf{U}_1) \cdot \mathbf{I}_\tau - (\mathbf{A}_\tau \cdot \mathbf{K} \cdot \mathbf{U}_1)\, \mathbf{n}$ for an arbitrary tangent vector $\mathbf{A}_\tau$!]. The first equation of the system (17) is decomposed as follows:

/IV,35

$$\left\{ \begin{aligned} &\boxed{c_0^2 \frac{\partial \rho_1}{\partial x_1} - \rho_0 c_0 \frac{\partial V_1}{\partial x_1} + g_{s_0} \frac{\partial s_1}{\partial x_1}} + \rho_0 \left(\frac{\partial V_1}{\partial x_0} + \mathbb{U}_1 \cdot \mathbb{D} c_0 + \mathbb{U}_1 \cdot (\mathbb{D}\mathbb{V}_0) \cdot \mathbb{n} \right) + \\ &\quad + \rho_0 x_1 \left(\mathbb{D} c_0 + (\mathbb{D}\mathbb{V}_0) \cdot \mathbb{n} \right) \cdot \mathbb{H} \cdot \left(\mathbb{D} V_1 - \mathbb{K} \cdot \mathbb{U}_1 \right) + \rho_1 \frac{\partial c_0^2}{\partial x_1} + s_1 \frac{\partial g_{s_0}}{\partial x_1} \\ &\quad - \frac{\rho_1}{\rho_0} \left(c_0^2 \frac{\partial \rho_0}{\partial x_1} + g_{s_0} \frac{\partial s_0}{\partial x_1} \right) + \rho_0 V_1 \frac{\partial \mathbb{V}_0}{\partial x_1} \cdot \mathbb{n} + \rho_0 \mathbb{U}_1 \cdot \mathbb{H} \cdot (\mathbb{D}\mathbb{V}_0) \cdot \mathbb{n} = 0, \\ &\boxed{\rho_0 c_0 \frac{\partial \mathbb{U}_1}{\partial x_1}} + \rho_0 V_1 \left(\mathbb{D} c_0 + (\mathbb{D}\mathbb{V}_0) \cdot \mathbb{n} \right) - \rho_0 \frac{\partial \mathbb{U}_1}{\partial x_0} \cdot \mathbb{I}_\tau - c_0^2 \mathbb{H} \cdot \mathbb{D} \rho_1 - g_{s_0} \mathbb{H} \cdot \mathbb{D} s_1 \\ &\quad - \rho_0 x_1 \left(\mathbb{D} c_0 + (\mathbb{D}\mathbb{V}_0) \cdot \mathbb{n} \right) \cdot \mathbb{H} \cdot \left(V_1 \mathbb{K} + (\mathbb{D}\mathbb{U}_1) \cdot \mathbb{I}_\tau \right) - \rho_1 \mathbb{H} \cdot \mathbb{D} c_0^2 - s_1 \mathbb{H} \cdot \mathbb{D} g_{s_0} + \\ &\quad + \frac{\rho_1}{\rho_0} \left(c_0^2\, \mathbb{H} \cdot \mathbb{D} \rho_0 + g_{s_0} \mathbb{H} \cdot \mathbb{D} s_0 \right) - \rho_0 V_1 \frac{\partial \mathbb{V}_0}{\partial x_1} \cdot \mathbb{I}_\tau - \rho_0 \mathbb{U}_1 \cdot \mathbb{H} \cdot (\mathbb{D}\mathbb{V}_0) \cdot \mathbb{I}_\tau = 0, \end{aligned} \right. \tag{30}$$

By forming the obvious combination of eqs.(17) and (30), we can obtain an equa-

tion containing none of the derivatives $\frac{\partial \rho_1}{\partial x_1}$, $\frac{\partial S_1}{\partial x_1}$, $\frac{\partial V_1}{\partial x_1}$, $\frac{\partial \mathbb{U}_1}{\partial x_1}$, which should be substituted for the first equation of the system (30). After these various operations, we obtain the following system:

$$
\left\{
\begin{aligned}
&\boxed{c_0 \frac{\partial S_1}{\partial x_1}} - \frac{\partial S_1}{\partial x_0} - V_1 \frac{\partial S_0}{\partial x_1} - \mathbb{U}_1 \cdot \mathbb{H} \cdot \mathbb{D} S_0 - x_1 \left(\mathbb{D} c_0 + (\mathbb{D} \mathbb{V}_0) \cdot n \right) \mathbb{H} \cdot \mathbb{D} S_1 = 0 \\
&\boxed{\frac{\partial}{\partial x_1} (\rho_0 V_1 - c_0 \rho_1)} + \frac{\partial \rho_1}{\partial x_0} + \rho_1 \frac{\partial c_0}{\partial x_1} + \rho_0 V_1 \mathbb{H} : \mathbb{K} + \mathbb{U}_1 \cdot \mathbb{H} \cdot \mathbb{D} \rho_0 + \rho_1 \nabla \cdot \mathbb{V}_0 \\
&\qquad + x_1 \left(\mathbb{D} c_0 + (\mathbb{D} \mathbb{V}_0) \cdot n \right) \cdot \mathbb{H} \cdot \mathbb{D} \rho_1 = 0 \\
&\boxed{\rho_0 c_0 \frac{\partial \mathbb{U}_1}{\partial x_1}} + \rho_0 V_1 \left(\mathbb{D} c_0 + (\mathbb{D} \mathbb{V}_0) \cdot n - \frac{\partial \mathbb{V}_0}{\partial x_1} \cdot \mathbb{I}_\pi \right) + \rho_1 \mathbb{H} \cdot \mathbb{D} c_0^2 - S_1 \mathbb{H} \cdot \mathbb{D} g_{s_0} + \\
&\qquad + \frac{\rho_1}{\rho_0} \left(c_0^2 \mathbb{H} \cdot \mathbb{D} \rho_0 + g_{s_0} \mathbb{H} \cdot \mathbb{D} S_0 \right) - c_0^2 \mathbb{H} \cdot \mathbb{D} \rho_1 - g_{s_0} \mathbb{H} \cdot \mathbb{D} S_1 \\
&\qquad - \rho_0 \frac{\partial \mathbb{U}_1}{\partial x_0} \cdot \mathbb{I}_\pi - \rho_0 \mathbb{U}_1 \cdot \mathbb{H} \cdot (\mathbb{D} \mathbb{V}_0) \cdot \mathbb{I}_\pi \\
&\qquad - \rho_0 x_1 \left(\mathbb{D} c_0 + (\mathbb{D} \mathbb{V}_0) \cdot n \right) \cdot \mathbb{H} \cdot \left(V_1 \mathbb{K} + (\mathbb{D} \mathbb{U}_1) \cdot \mathbb{I}_\pi \right) = 0 \qquad (31) \\
&\rho_0 c_0 \frac{\partial V_1}{\partial x_0} + c_0^2 \frac{\partial \rho_1}{\partial x_0} + g_{s_0} \frac{\partial S_1}{\partial x_0} + \\
&\quad + V_1 \left(\rho_0 c_0 \frac{\partial \mathbb{V}_0}{\partial x_1} \cdot n + \rho_0 c_0^2 \mathbb{H} \cdot \mathbb{K} + c_0^2 \frac{\partial \rho_0}{\partial x_1} + g_{s_0} \frac{\partial S_0}{\partial x_1} \right) \\
&\quad - \frac{\rho_1}{\rho_0} \left(c_0^3 \frac{\partial \rho_0}{\partial x_1} + c_0 g_{s_0} \frac{\partial S_0}{\partial x_1} \right) + c_0^2 \rho_1 \nabla \cdot \mathbb{V}_0 + c_0 \rho_1 \frac{\partial c_0^2}{\partial x_1} + c_0 S_1 \frac{\partial g_{s_0}}{\partial x_1} \\
&\quad + \mathbb{U}_1 \cdot \left\{ \rho_0 c_0 \left(\mathbb{D} c_0 + (\mathbb{D} \mathbb{V}_0) \cdot n + \mathbb{H} \cdot (\mathbb{D} \mathbb{V}_0) \cdot n \right) + c_0^2 \mathbb{H} \cdot \mathbb{D} \rho_0 + g_{s_0} \mathbb{H} \cdot \mathbb{D} S_0 \right\} + \\
&\quad + \rho_0 c_0^2 (\mathbb{H} \cdot \mathbb{D}) \cdot \mathbb{U}_1 + \\
&\quad + x_1 \left\{ \rho_0 c_0 \left(\mathbb{D} c_0 + (\mathbb{D} \mathbb{V}_0) \cdot n \right) \cdot \mathbb{H} \cdot \left(\mathbb{D} V_1 - \mathbb{K} \cdot \mathbb{U}_1 \right) + \right. \\
&\qquad + c_0^2 \left(\mathbb{D} c_0 + (\mathbb{D} \mathbb{V}_0) \cdot n \right) \cdot \mathbb{H} \cdot \mathbb{D} \rho_1 \\
&\qquad \left. + g_{s_0} \left(\mathbb{D} c_0 + (\mathbb{D} \mathbb{V}_0) \cdot n \right) \cdot \mathbb{H} \cdot \mathbb{D} S_1 \right\} = 0 .
\end{aligned}
\right.
$$

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That we have lost the derivative in x_1 of the quantities of rank 1 in the last equation of the system (31) obviously is due to the fact that the surface $x_1 = 0$ represents, at each instant, the position of an acoustic wave surface, i.e., the intersection, at t = const, of a space-time characteristic surface.

4.2.3 Acoustic Field on an Acoustic Wave

We are now able to pick up the problem which we had indicated briefly in Section 4.1.6. Let us recall that we wanted to discuss the conditions under which an acoustic field can admit of a part concentrated on a surface. We have seen already that such a field must carry an acoustic or an entropy wave; on this basis, making use of the preceding Section, we will construct the formula complex adapted to the second case. Thus, let us examine the conditions under which the "functions" v_1, ρ_1, S_1, $\mathbb{U}_1$ of the preceding Section might comprise, in part, a distribution carried by $x_1 = 0$. Equations (31) clearly demon- /IV,37 strate what we already know: The distributions corresponding to v_1 and ρ_1 are of the same order, while those corresponding to S_1 and $\mathbb{U}_1$ are of a lower order. It is also obvious that, so far as v_1 and ρ_1 are concerned and if the highest possible order is assumed, we must have

$$\rho_0 V_1 - c_0 \rho_1 = 0, \qquad S_1 = 0 \tag{32}$$

constituting relations which we obtained and included already in eq.(1.110); however, at that time we still had an arbitrary quantity, namely that of the function ϕ. It is obvious that the last equation of the system (31) represents a condition imposed on the arbitrariness in question. In fact, still retaining the highest possible order, we can make use of eq.(32) and set $S_1 = 0$, $\mathbb{U}_1 = 0$ from which it follows that the last equation of the system (31) can be given the following form:

$$2\frac{\partial V_1}{\partial x_0} + V_1\left\{c_0 K + \frac{c_0}{\rho_0}\frac{\partial}{\partial x_0}\left(\frac{\rho_0}{c_0}\right) + \frac{1}{c_0}\frac{\partial c_0^2}{\partial x_1} + \nabla\cdot\mathbb{V}_0 + \frac{\partial \mathbb{V}_0}{\partial x_1}\cdot n\right\} = 0, \tag{33}$$

always retaining the distribution in x_1 of the highest possible order and noting

$$K = \frac{K_2}{H_2} + \frac{K_3}{H_3} = \mathbb{K} : \mathbb{I}_T, \tag{34}$$

which is the mean double curvature of the wave surface.

We will attempt to eliminate the derivative $\frac{\partial c_0^2}{\partial x_1}$, taking the following equations into consideration:

$$\begin{cases} \frac{\partial \rho_0}{\partial x_0} - c_0 \frac{\partial \rho_0}{\partial x_1} + \rho_0 \nabla\cdot\mathbb{V}_0 + O(x_1) = 0, \\ \frac{\partial S_0}{\partial x_0} - c_0 \frac{\partial S_0}{\partial x_1} + O(x_1) = 0, \end{cases} \tag{35}$$

which express the conservation of mass and entropy for an unperturbed fluid with characteristic variables. For an obvious combination of the two equations of the system (35), we obtain

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$$\frac{\partial c_0^2}{\partial x_0} - c_0 \frac{\partial c_0^2}{\partial x_1} + 2c_0^2 (\Gamma_0 - 1)\, \nabla \cdot \mathbb{V}_0 = 0\,(x_1), \tag{36}$$

where we note

$$\Gamma = \frac{1}{c}\left(\frac{\partial \rho c}{\partial \rho}\right)_{S = \text{const}} \tag{37}$$

It should be mentioned that, for an ideal gas, we have $\Gamma = \frac{\gamma + 1}{2}$. Substituting eq.(36) into eq.(33), we obtain

$$2\frac{\partial V_1}{\partial x_0} + V_1 \left\{ c_0 K + \frac{1}{\rho_0 c_0} \frac{\partial \rho_0 c_0}{\partial x_0} + (2\Gamma_0 - 1)\, \nabla \cdot \mathbb{V}_0 + \frac{\partial \mathbb{V}_0}{\partial x_1} \cdot n \right\} = 0 \tag{38}$$

for the highest-order component of the distribution V_1.

Theorem 10: The acoustic equations in a nonhomogeneous atmosphere without exterior energy supply* tolerate solutions that comprise distributions carried by the acoustic wave surfaces. Let x_1 be the distance from a point in space, at the instant t, to the wave surface at the same instant, so that the solution in question will obtain the following form:

$$\left\{ \begin{aligned} \mathbb{V}_1 &= c_0 \phi \frac{d^K \delta(x_1)}{d x_1^K}\, n + \cdots \\ \rho_1 &= \rho_0 \phi \frac{d^K \delta(x_1)}{d x_1^K} + \cdots \\ S_1 &= \cdots\cdots, \end{aligned} \right. \tag{39}$$

where the dots represent distributions of a lower order and where n denotes the unit vector normal to the surface of a wave directed in the sense of propagation. If $\frac{1}{2}K$ denotes the mean curvature of the wave surface, then the variation in ϕ along a sound ray will be governed by

$$\frac{1}{\rho_0 c_0^3 \phi^2} \frac{d}{dt}\left(\rho_0 c_0^3 \phi^2\right) + c_0 K + (2\Gamma_0 - 1)\, \nabla \cdot \mathbb{V}_0 + n \cdot (\nabla \mathbb{V}_0) \cdot n = 0. \tag{40}$$

---

* This restriction is not essential, since one could just as well lift it by returning to the entropy equation.

4.2.4 Energy Considerations

If we write

$$\left\{\begin{aligned} V_1 &= V_1' \frac{d^\kappa \delta}{dx_1^\kappa} + \dots, \\ \rho_1 &= \rho_1' \frac{d^\kappa \delta}{dx_1^\kappa} + \dots, \\ s_1 &= \dots, \end{aligned}\right. \tag{41}$$

it becomes obvious that the correlation between ρ_1' and $\mathbf{V}_1'$ is the same as in a plane wave. In addition, using ρ_1' and $\mathbf{V}_1'$ which are considered as constituting an acoustic field, the following can be formed:

Volume density of acoustic energy:

$$\mathcal{E}' = \rho_0 c_0^2 \phi^2, \tag{42}$$

Surface density vector of the acoustic energy flux:

$$\mathbf{W}' = \mathcal{E}'(\mathbf{V}_0 + c_0 \mathbf{n}), \tag{43}$$

Volume density of the acoustic energy production:

$$\Pi' = \mathcal{E}'\left\{ \mathbf{n}\cdot(\nabla \mathbf{V}_0)\cdot\mathbf{n} + (\Gamma_0 - 1)\, \nabla\cdot\mathbf{V}_0 \right\}. \tag{44}$$

To obtain eq.(44), it must be noted that eq.(11) is reduced to

$$\begin{aligned} \Pi' = {}& \rho_0 V_1'^2\, \mathbf{n}\cdot(\nabla \mathbf{V}_0)\cdot\mathbf{n} + \left\{ \frac{\rho_1'^2}{\rho_0} c_0^2 \left(1 - \frac{p_0}{\rho_0 c_0^2}\right) - \right. \\ & \left. - \frac{\rho_0 \rho_1'^2}{2} \frac{\partial}{\partial \rho_0}\left[\frac{1}{\rho_0}\left(\frac{p_0}{\rho_0} - c_0^2\right)\right]\right\} \nabla\cdot\mathbf{V}_0 + \\ & + \frac{\rho_1'^2}{2}\left\{ \frac{\partial}{\partial s_0}\left[\frac{1}{\rho_0}\left(\frac{p_0}{\rho_0} - c_0^2\right)\right] + \frac{T_0}{\rho_0}\left(1 - \frac{2\,\partial \rho_0}{\rho_0 T_0}\right)\left(1 + \frac{\rho_0}{T_0}\frac{\partial T_0}{\partial \rho_0}\right)\right\}\left(\frac{\partial S_0}{\partial t} + \mathbf{V}_0\cdot\nabla S_0\right), \end{aligned} \tag{45}$$

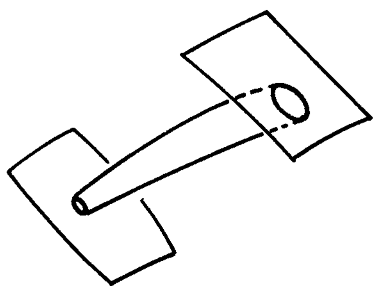
but that $\frac{\partial S_0}{\partial t} + \mathbf{V}_0 \cdot \nabla S_0$ is neglected here, yielding

$$1 - \frac{p_0}{\rho_0 c_0^2} - \frac{\rho_0^2}{2c_0^2}\frac{\partial}{\partial \rho_0}\left[\frac{1}{\rho_0}\left(\frac{p_0}{\rho_0} - c_0^2\right)\right] = \Gamma_0 - 1 \,. \tag{46}$$

Under these conditions, returning to eq.(40), it is obvious that this equation is interpreted as an equation of acoustic energy evolution along the sound ray, i.e.,

$$\frac{d\, c_0 \mathcal{E}'}{d\, t} + c_0 \mathcal{E}'\left\{ c_0 K + \Gamma_0 \nabla.\mathbb{V}_0 \right\} + c_0 \Pi' = 0 \,. \tag{47}$$

We will attempt to give a different interpretation to this equation. For this, we will define, on each sound ray, a function $\mathfrak{A}(x_0)$ which is assumed to be



proportional to the area of the intersection of the wave by an infinitely delimited stream tube of sound rays. Let us return to the representation (1) and let

$$\mathfrak{a} = (a_1, a_2) \,, \tag{48}$$

be a parametric representation of the initial position of the point $\mathbb{P}$, yielding

$$d\mathbb{P} = \left(\mathbb{V}_0 + c_0 \mathfrak{n}\right)\cdot\left(\frac{\partial \mathbb{P}}{\partial a_1} \wedge \frac{\partial \mathbb{P}}{\partial a_2}\right) da_1\, da_2\, dx_0 \,, \tag{49}$$

such that, for $\mathfrak{A}(x_0;\, a_1, a_2)$, the following formula can be adopted:

$$\mathcal{A}(x_0;\, a_1, a_2) = f(a_1, a_2)\;\; \mathfrak{n}\cdot\left(\frac{\partial \mathbb{P}}{\partial a_1} \wedge \frac{\partial \mathbb{P}}{\partial a_2}\right), \tag{50}$$

where the function f remains arbitrary. We have

$$\frac{1}{\mathcal{A}}\frac{\partial \mathcal{A}}{\partial x_0} = \frac{\mathbb{n}\cdot\left\{\frac{\partial \mathbb{V}_0 + c_0\mathbb{n}}{\partial a_1}\wedge\frac{\partial \mathbb{P}}{\partial a_2}\right\} + \mathbb{n}\cdot\left\{\frac{\partial \mathbb{P}}{\partial a_1}\wedge\frac{\partial \mathbb{V}_0 + c_0\mathbb{n}}{\partial a_2}\right\}}{\mathbb{n}\cdot\left(\frac{\partial \mathbb{P}}{\partial a_1}\wedge\frac{\partial \mathbb{P}}{\partial a_2}\right)}, \tag{51}$$

so that it is obvious that the right-hand side does not depend on the selected coordinates (a_1, a_2). We leave to the reader the task of proving this point. Therefore, it is preferable to select, for a_1 and a_2, the coordinates x_2 and x_3 on $\Sigma(t)$ itself, such as we had introduced in Section 4.2.1, which will then yield /IV,41

$$\frac{1}{\mathcal{A}}\frac{\partial \mathcal{A}}{\partial x_0} = \frac{\mathbb{n}\cdot\left(\frac{\partial \mathbb{V}_0 + c_0\mathbb{n}}{H_2\,\partial x_2}\wedge\mathbb{E}_3\right) + \mathbb{n}\cdot\left(\mathbb{E}_2\wedge\frac{\partial \mathbb{V}_0 + c_0\mathbb{n}}{H_3\,\partial x_3}\right)}{\mathbb{n}\cdot\left(\mathbb{E}_2\wedge\mathbb{E}_3\right)}, \tag{52}$$

where the denominator is equal to unity. From this, we obtain

$$\begin{aligned}\frac{1}{\mathcal{A}}\frac{\partial \mathcal{A}}{\partial x_0} &= \mathbb{E}_2\cdot\left\{\nabla\left(\mathbb{V}_0 + c_0\mathbb{n}\right)\right\}\cdot\mathbb{E}_2 + \mathbb{E}_3\cdot\left\{\nabla\left(\mathbb{V}_0 + c_0\mathbb{n}\right)\right\}\cdot\mathbb{E}_3 \\ &= \nabla\cdot\mathbb{V}_0 - \mathbb{n}\cdot\left(\nabla\mathbb{V}_0\right)\cdot\mathbb{n} + c_0 K\end{aligned} \tag{53}$$

such that eq.(47) will have the following form:

$$\begin{aligned}\frac{d c_0\mathcal{E}'\mathcal{A}}{dt} &+ c_0\mathcal{E}'\mathcal{A}\left\{\mathbb{n}\cdot\left(\nabla\mathbb{V}_0\right)\cdot\mathbb{n} + (\Gamma_0 - 1)\nabla\cdot\mathbb{V}_0\right\} + \\ &+ c_0\Pi'\mathcal{A} = 0 .\end{aligned} \tag{54}$$

It is necessary to transform this equation by posing

$$W_0 = c_0 + \mathbb{V}_0\cdot\mathbb{n} = c_0 + u_0 \tag{55}$$

for the velocity of normal displacement of the wave. This will furnish

$$\begin{aligned}\frac{d\, W_0\mathcal{E}'\mathcal{A}}{dt} + W_0\Pi'\mathcal{A} = \mathcal{A}\mathcal{E}'\Big\{&-\frac{u_0}{c_0}\frac{\partial c_0}{\partial x_0} + \frac{\partial u_0}{\partial x_0} - \\ &- W_0\left[\mathbb{n}\cdot\left(\nabla\mathbb{V}_0\right)\cdot\mathbb{n} + (\Gamma_0 - 1)\nabla\cdot\mathbb{V}_0\right]\Big\},\end{aligned} \tag{56}$$

where we will transform the expression in braces. Let us note first that we have

$$\frac{\partial}{\partial x_0} = \frac{\partial}{\partial t} + (\mathbf{V}_0 + c_0 \mathbf{n})\cdot\nabla = \frac{D_0}{Dt} + c_0\, \mathbf{n}\cdot\nabla , \tag{57}$$

from which it follows that

/IV,42

$$\begin{aligned}
\{\ \} &= \frac{D_0 \mathbf{V}_0}{Dt}\cdot\mathbf{n} + c_0\, \mathbf{n}\cdot(\nabla \mathbf{V}_0)\cdot\mathbf{n} + \mathbf{V}_0\cdot\frac{\partial \mathbf{n}}{\partial x_0} - \\
&\quad - (\Gamma_0 - 1)\frac{u_0}{\rho_0}\frac{D_0\rho_0}{Dt} - u_0\, \mathbf{n}\cdot\nabla c_0 - u_0\left\{\mathbf{n}\cdot(\nabla\mathbf{V}_0)\cdot\mathbf{n} + (\Gamma_0 - 1)\nabla\cdot\mathbf{V}_0\right\} \\
&= \frac{\partial \mathbf{V}_0}{\partial t}\cdot\mathbf{n} + (\mathbf{V}_0 - u_0\mathbf{n})\cdot(\nabla\mathbf{V}_0)\cdot\mathbf{n} + \mathbf{V}_0\cdot\frac{\partial\mathbf{n}}{\partial x_0} \\
&\qquad - u_0\,\mathbf{n}\cdot\nabla c_0 - (\Gamma_0 - 1)\, c_0\, \nabla\cdot\mathbf{V}_0 \\
&= \frac{\partial \mathbf{V}_0}{\partial t}\cdot\mathbf{n} + \frac{\partial c_0}{\partial t} - \frac{D_0 c_0}{Dt} - (\Gamma_0 - 1)\, c_0\, \nabla\cdot\mathbf{V}_0 \\
&= \frac{\partial \mathbf{V}_0}{\partial t}\cdot\mathbf{n} + \frac{\partial c_0}{\partial t} ,
\end{aligned} \tag{58}$$

where we have used

$$\left\{\begin{aligned}
&\frac{D_0 S_0}{Dt} = \frac{\partial S_0}{\partial t} + \mathbf{V}_0\cdot\nabla S_0 = 0 \\
&\frac{\partial c_0}{\partial \rho_0} = \frac{c_0}{\rho_0}(\Gamma_0 - 1)
\end{aligned}\right. \tag{59}$$

and

$$-\frac{\partial \mathbf{n}}{\partial x_0} = \nabla_T c_0 + (\nabla_T \mathbf{V}_0)\cdot\mathbf{n} \qquad \nabla_T = \nabla - \mathbf{n}\,\mathbf{n}\cdot\nabla \tag{60}$$

<u>Theorem 11</u>: If a sound field comprises a concentrated acoustic wave

$$\mathbf{V}_1 = \mathbf{V}_1' \frac{d^\kappa \delta}{dx_1^\kappa} + \cdots \tag{61}$$

$$\begin{cases} \rho_1 = \rho_1' \dfrac{d^K \rho}{dx_1^K} + \dots \\ S_1 = \dots\dots \end{cases}$$

with the relation

$$\mathbb{V}_1' = \frac{c_0}{\rho_0} \rho_1' \, \mathbb{n} \,, \tag{62}$$

where $\mathbb{n}$ is the unit vector normal to the wave surface directed in the /IV,43
sense of propagation and if, with ρ_1' and $\mathbb{V}_1'$, we form the volume density of acoustic energy

$$\mathcal{E}' = \frac{1}{2}\left(\rho_0 |\mathbb{V}_1'|^2 + c_0^2 \frac{\rho_1'^2}{\rho_0}\right) = \rho_0 |\mathbb{V}_1'|^2 = \frac{c_0^2 \rho_1'^2}{\rho_0}, \tag{63}$$

and the surface density vector of the acoustic energy flux

$$\mathbb{W}' = \mathcal{E}' (\mathbb{V}_0 + c_0 \mathbb{n}) \,, \tag{64}$$

then the evolution of eq.(61) in the course of time, following a sound ray, will be ruled by the differential equation

$$\frac{d}{dt}(A \, \mathbb{n}.\mathbb{W}') + A \, \Pi'.(\mathbb{V}_0 + c\mathbb{n}) - A \, \mathcal{E}'\left(\frac{\partial \mathbb{V}_0}{\partial t}.\mathbb{n} + \frac{\partial c_0}{\partial t}\right) = 0, \tag{65}$$

denoting here

$$\Pi' = \Pi' \, \mathbb{n}, \tag{66}$$

with

$$\Pi' = \mathcal{E}' \left\{ \mathbb{n}.(\nabla \mathbb{V}_0).\mathbb{n} + (\Gamma_0 - 1) \, \nabla . \mathbb{V}_0 \right\} , \tag{67}$$

in such a manner that Π' will be the volume density of the acoustic energy production corresponding to the acoustic field ρ_1', $\mathbb{V}_1'$, $S_1' = 0$. This theorem holds only if it is assumed that the nonperturbed atmosphere contains no dissipative phenomena, i.e., if

$$\frac{\partial S_0}{\partial t} + \mathbb{V}_0 . \nabla S_0 = 0 \,. \tag{68}$$

In eq.(65), $\mathfrak{A}$ denotes a function which varies, along the sound ray, proportionally to the cross-sectional area of a tube of rays infinitely unbounded by the wave surface.

The presence of the last term in eq.(65) may seem uncommon in that /IV,44 it destroys the elegance of the result. Actually, this is by no means so and, if the result is not as compact as one could wish, this is due to the fact that the selection of the function $\mathcal{A}$ is not quite as fortunate as would appear at first. In this respect, we will find Section 4.1.4 highly useful. Let $\psi(x_0, \boldsymbol{x}) = 0$ be the equation of the wave surface (for each x_0) and let us use again the notations (1.45). To define the point instants, we can use the coordinates a_1, a_2, x_1 which had been introduced previously and then can substitute ψ for x_0. Under these conditions, the volume element in the four-dimensional space of the point instants will become

$$\frac{1}{f(a_1, a_2)} \frac{\mathcal{A}(x_0; a_1, a_2)}{\left|\frac{\partial \psi}{\partial x_0}\right|} \, da_1 \, da_2 \, dx_1 \, d\psi = dx_0 \, d\boldsymbol{x} . \tag{69}$$

Thus, considering the function

$$\frac{\mathcal{A}}{|p_0|} = \mathcal{C} , \tag{70}$$

eq.(65) can be written as follows:

$$\frac{d}{dt}\left(\mathcal{C}\, \boldsymbol{n} \cdot \boldsymbol{W}'\right) + \mathcal{C}\, \Pi' \cdot (\boldsymbol{V}_0 + c_0 \boldsymbol{n}) + \mathcal{C}\, \xi' \left\{ \frac{1}{|\boldsymbol{p}|} \frac{d|\boldsymbol{p}|}{dt} - \right.$$
$$\left. - \frac{\partial \boldsymbol{V}_0}{\partial t} \cdot \boldsymbol{n} - \frac{\partial c_0}{\partial t} \right\} = 0 . \tag{71}$$

Along the wave, we have

$$p_0 + \boldsymbol{V}_0 \cdot \boldsymbol{p} = c_0 |\boldsymbol{p}| , \tag{72}$$

such that, if we select

$$\boldsymbol{p} = - |\boldsymbol{p}| \, \boldsymbol{n} \tag{73}$$

as had been done previously, we will obtain

/IV,45

$$p_0 = (c_0 + \boldsymbol{V}_0 \cdot \boldsymbol{n}) |\boldsymbol{p}| = |p_0| . \tag{74}$$

Then, eq.(1.53) shows that, along the sound ray, we have

$$\frac{1}{p}\frac{dp_0}{dt} = \frac{\partial c_0}{\partial t} + \frac{\partial \mathbb{V}_0}{\partial t} \cdot \mathbb{n}, \tag{75}$$

in such a manner that eq.(71) is elegantly reduced to

$$\frac{d\, \mathcal{G}\, \mathbb{n}.\mathbb{W}'}{dt} + \mathcal{G}\, \Pi' \cdot (\mathbb{V}_0 + c_0 \mathbb{n}) = 0 . \tag{76}$$

Theorem 12: The conditions are those of theorem 11, in addition to which the differential system defining the sound rays is considered:

$$\begin{cases} \dfrac{d\mathbb{P}}{dt} = (c_0 + \mathbb{V}_0 . \mathbb{n})|p_0| \\ \dfrac{1}{|p|}\dfrac{d\, p\, \mathbb{n}}{dt} = -(\nabla \mathbb{V}_0) . \mathbb{n} - \nabla c_0 . \end{cases} \tag{77}$$

We then form the expression ($p > 0$)

$$\mathcal{G} = \frac{\mathcal{A}}{|p|(c_0 + \mathbb{V}_0 . \mathbb{n})}, \tag{78}$$

whose significance is the following: Let $\psi(x_0, \mathbf{x}) = 0$ be the equation of the wave surface and let $\mathbb{P}(a_1, a_2, x_0)$ be a parametric representation of the latter in such a manner that a_1 and a_2 remain constant over the sound rays; then, there exists a function depending only on a_1 and a_2 such that the four-dimensional volume element, in the neighborhood of the wave, will be expressed in the following form:

$$dx_0\, d\mathbf{x} = \frac{\mathcal{G}(x_0; a_1, a_2)}{\rho(a_1; a_2)}\, da_1\, da_2\, dx_1\, d\psi, \tag{79}$$

where x_1 denotes the (orthogonal) distance to the wave surface (at fixed x_0). With these notations, the equation for evaluating the acoustic field centered on the wave surface can be written as follows: /IV,46

$$\frac{d}{dt}(\mathcal{G}\,\mathbb{W}' . \mathbb{n}) + \mathcal{G}\, \Pi' \cdot (\mathbb{V}_0 + c_0 \mathbb{n}) = 0 . \tag{80}$$

4.2.5 Wave Trains of Small Width

Let us first assume that the atmosphere is homogeneous and calm and let us write eqs.(31) for this case; we then obtain

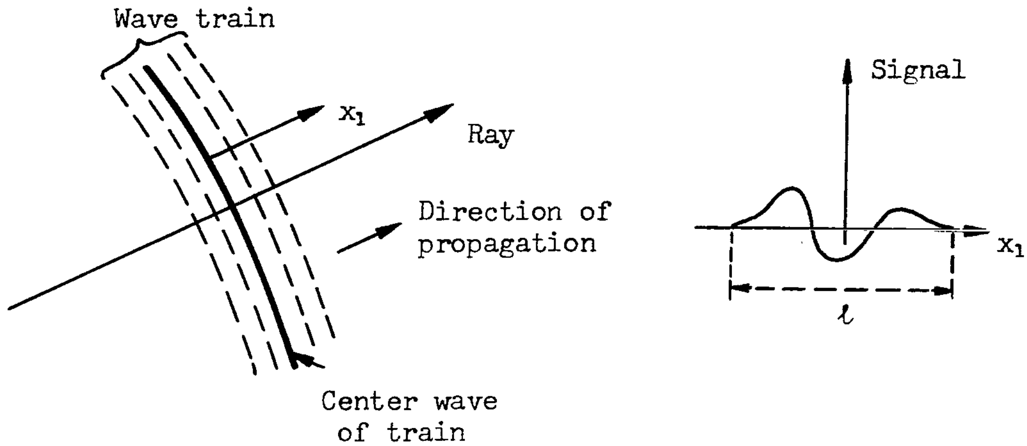
$$\left\{\begin{aligned}
&\boxed{c_0 \frac{\partial s_1}{\partial x_1}} - \frac{\partial s_1}{\partial x_0} = 0 , \\
&\boxed{\frac{\partial}{\partial x_1}(\rho_0 V_1 - c_0 \rho_1)} + \frac{\partial \rho_1}{\partial x_0} + \rho_0 V_1 \, \mathbb{H} : \mathbb{K} = 0 , \\
&\boxed{\rho_0 c_0 \frac{\partial \mathbb{U}_1}{\partial x_1}} - c_0^2 \, \mathbb{H} \cdot \mathbb{D} \rho_1 - g_{s_2} \mathbb{H} \cdot \mathbb{D} s_1 - \rho_0 \frac{\partial \mathbb{U}}{\partial x_0} \cdot \mathbb{I}_T = 0 , \\
&\rho_0 c_0 \frac{\partial V_1}{\partial x_0} + c_0^2 \frac{\partial \rho_1}{\partial x_0} + g_{s_0} \frac{\partial s_1}{\partial x_0} + \rho_0 c_0^2 V_1 \mathbb{K} : \mathbb{H} + \rho_0 c_0^2 (\mathbb{H} \cdot \mathbb{D}) \cdot \mathbb{U}_1 = 0 .
\end{aligned}\right. \tag{81}$$

It will be recalled that

$$\left\{\begin{aligned}
&\mathbb{H} = \frac{H_2}{H_2 + x_1 K_2} \mathbb{e}_2 \mathbb{e}_2 + \frac{H_3}{H_3 + x_1 K_3} \mathbb{e}_3 \mathbb{e}_3 , \\
&\mathbb{D} = \frac{\mathbb{e}_2}{H_2} \frac{\partial}{\partial x_2} + \frac{\mathbb{e}_3}{H_3} \frac{\partial}{\partial x_3} , \\
&\mathbb{K} = \frac{K_2}{H_2} \mathbb{e}_2 \mathbb{e}_2 + \frac{K_3}{H_3} \mathbb{e}_3 \mathbb{e}_3 ,
\end{aligned}\right. \tag{82}$$

where x_2 and x_3 - for each x_0 - are curvilinear coordinates selected in /IV,47 such a manner that the coordinate lines will be the curvature lines on the wave surface $x_1 = 0$.

Let us now imagine a situation in which the acoustic field ρ_1, V_1, $\mathbb{U}_1 S_1$ becomes very small outside of a zone having a width $O(\ell)$ spread along the wave surface and let us assume that the latter, since its generation, had traveled over a distance $O(L)$; we then will attempt to construct an asymptotic representation of this acoustic field when $\frac{\ell}{L} \ll 1$. Such is the case under the conditions given in the accompanying diagram; we wish to obtain a proximal representation in the terminology of Chapter I. For this purpose, let us introduce a nondimensionality, by posing /IV,48



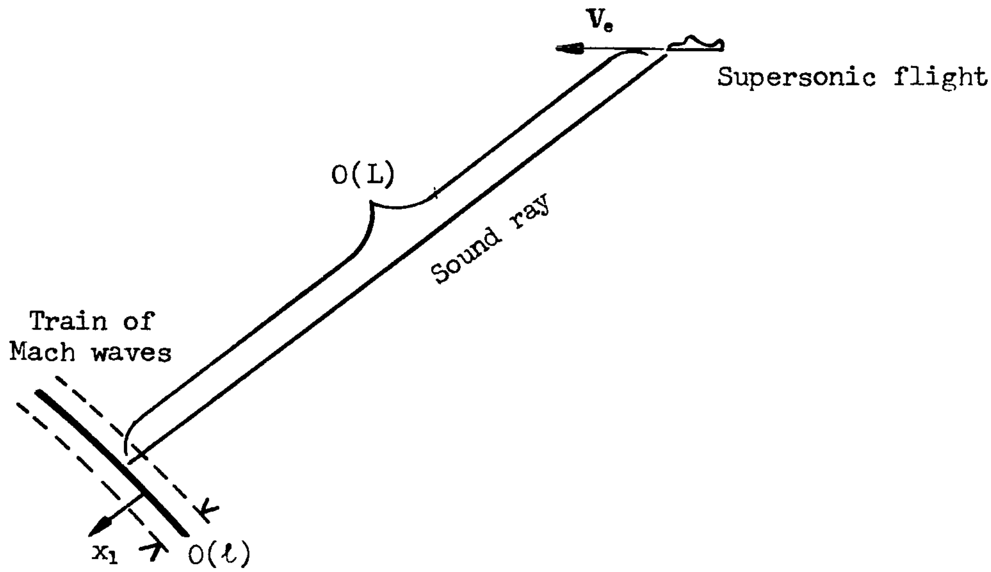
$$
\left\{
\begin{array}{l}
V_1 = c_0\,\bar{u}\,, \quad p_1 = \rho_0 c_0^2\,\bar{p}\,, \quad \rho_1 = \rho_0\,\bar{\rho}\,, \quad S_1 = C_v\,\bar{s} \\
\mathbb{U}_1 = c_0\,\bar{\mathbb{v}}_\tau\,, \quad x_1 = \ell\,\bar{x}_1\,, \quad x_0 = L\,c_0^{-1}\,\bar{x}_0\,, \\
x_2 = L\,\bar{x}_2\,, \quad x_3 = L\,\bar{x}_3\,, \quad K_2 = L^{-1}\bar{K}_2\,, \quad K_3 = L^{-1}\bar{K}_3\,, \\
H_2 = \bar{H}_2\,, \quad H_3 = \bar{H}_3\,, \quad \mathbb{K} = L^{-1}\,\bar{\mathbb{K}}\,, \quad \mathbb{D} = L^{-1}\,\bar{\mathbb{D}}\,, \\
g_{s_0}\,c_v = \rho_0\,c_0^2\,\varpi .
\end{array}
\right. \tag{83}
$$

and introducing an infinitely small principal term

$$
\varepsilon = \frac{\ell}{L} \ll 1 . \tag{84}
$$

Then, eqs.(81) will be written as

$$
\left\{
\begin{array}{l}
\boxed{\dfrac{\partial \bar{s}}{\partial \bar{x}_1}} - \varepsilon\,\dfrac{\partial \bar{s}}{\partial \bar{x}_0} = 0\,, \\
\boxed{\dfrac{\partial\,\bar{u} - \bar{\rho}}{\partial \bar{x}_1}} + \varepsilon\left\{\dfrac{\partial \bar{\rho}}{\partial \bar{x}_0} + \bar{u}\;\bar{\mathbb{K}} : \left(\mathbb{I}_\tau + \displaystyle\sum_{n=1}^{\infty} \varepsilon^n \bar{x}_1^n \bar{\mathbb{H}}_n\right)\right\} = 0\,, \\
\boxed{\dfrac{\partial \bar{\mathbb{v}}_\tau}{\partial \bar{x}_1}} - \varepsilon\left\{\left(\mathbb{I}_\tau + \displaystyle\sum_{n=1}^{\infty} \varepsilon^n \bar{x}_1^n \bar{\mathbb{H}}_n\right) \cdot \left(\bar{\mathbb{D}}\,\bar{\rho} + \varpi\,\bar{\mathbb{D}}\,\bar{s}\right) + \dfrac{\partial \bar{\mathbb{v}}_\tau}{\partial \bar{x}_0} \cdot \mathbb{I}_\tau\right\} = 0
\end{array}
\right. \tag{85}
$$



$$\frac{\partial \bar{u}}{\partial \bar{x}_0} + \frac{\partial \bar{\rho}}{\partial \bar{x}_0} + \varpi \frac{\partial \bar{s}}{\partial \bar{x}_0} + \left(\bar{\mathbb{I}}_T + \sum_{n=1}^{\infty} \varepsilon^n \bar{x}_1^{\,n} \bar{\mathbb{H}}_n\right) : \bar{\mathbb{K}}\ \bar{u} + \left\{\left(\mathbb{I}_\varphi + \sum_{n=1}^{\infty} \varepsilon^n \bar{x}_1^{\,n} \bar{\mathbb{H}}_n\right) \cdot \bar{\mathbb{D}}\right\} \cdot \bar{\mathbb{V}}_\varphi = 0 \tag{86}$$

after having noted that

$$\begin{aligned} \mathbb{H} &= \frac{H_2}{H_2 + \varepsilon \bar{x}_2 \bar{K}_2}\, \mathbb{e}_2 \mathbb{e}_2 + \frac{H_3}{H_3 + \varepsilon \bar{x}_3 \bar{K}_3}\, \mathbb{e}_3 \mathbb{e}_3 \\ &= \mathbb{e}_2 \mathbb{e}_2 + \mathbb{e}_3 \mathbb{e}_3 + \sum_{n=1}^{\infty} \varepsilon^n \bar{x}_1^{\,n} \bar{\mathbb{H}}_n \,. \end{aligned} \tag{87}$$

The form of eqs.(85) and (86) suggests searching for the solution in the form of an asymptotic expansion /IV,49

$$\begin{cases} \bar{u} \simeq \sum_{n=0}^{\infty} \varepsilon^n \bar{u}_n \\ \bar{\rho} \simeq \sum_{n=0}^{\infty} \varepsilon^n \bar{\rho}_n \end{cases} \tag{88}$$

$$\left\{\begin{array}{l}\bar{s} \cong \sum_{n=0}^{\infty} \varepsilon^{n} \bar{s}_{n} \\ \bar{V}_{\tau} \cong \sum_{n=0}^{\infty} \varepsilon^{n} \bar{V}_{\tau n}\end{array}\right.$$

leading, for quantities of rank 0, 1, ..., n, ... to a system hierarchy of highly decoupled differentials.

Systems of rank zero ($\bar{K} = \mathbb{I}_T : \mathbb{K}$)

$$\left\{\begin{array}{l}\dfrac{\partial \bar{s}_0}{\partial \bar{x}_1} = 0 , \\ \dfrac{\partial \bar{u}_0 - \bar{\rho}_0}{\partial \bar{x}_1} = 0 , \\ \dfrac{\partial \bar{V}_{\tau 0}}{\partial \bar{x}_1} = 0 ,\end{array}\right. \tag{89}$$

$$\frac{\partial \bar{u}_0}{\partial \bar{x}_0} + \frac{\partial \bar{\rho}_0}{\partial \bar{x}_0} + \varpi \frac{\partial \bar{s}_0}{\partial \bar{x}_0} + \bar{K}\, \bar{u}_0 + \bar{\mathbb{D}} \cdot \bar{V}_{\tau_0} = 0 . \tag{90}$$

System of rank n:

$$\left\{\begin{array}{l}\dfrac{\partial \bar{s}_n}{\partial \bar{x}_1} = \dfrac{\partial \bar{s}_{n-1}}{\partial \bar{x}_1} \\ \dfrac{\partial \bar{u}_n - \bar{\rho}_n}{\partial \bar{x}_1} = \dfrac{\partial \bar{\rho}_{n-1}}{\partial \bar{x}_0} + \bar{u}_{n-1} \bar{K} + \sum_{p=2}^{n} \bar{u}_{n-p}\, \bar{x}_1^{\,p}\, \mathbb{K} : \mathbb{H}_p \\ \dfrac{\partial \bar{V}_{\tau n}}{\partial \bar{x}_1} = \dfrac{\partial \bar{V}_{\tau n-1}}{\partial \bar{x}_0} \cdot \mathbb{I}_\tau + \bar{\mathbb{D}}\, \bar{\rho}_{n-1} + \bar{\varpi}\, \bar{\mathbb{D}}\, \bar{s}_{n-1} + \\ \qquad + \sum_{p=2}^{n} \bar{x}_1^{\,p}\, \mathbb{H}_p \cdot \left(\bar{\mathbb{D}} \bar{\rho}_{n-p} + \varpi\, \bar{\mathbb{D}}\, \bar{s}_{n-p} \right)\end{array}\right. \tag{91}$$

/IV,50

$$\frac{\partial \bar{u}_n}{\partial \bar{x}_0} + \frac{\partial \bar{\rho}_n}{\partial \bar{x}_0} + \varpi \frac{\partial \bar{s}_n}{\partial \bar{x}_0} + K\, \bar{u}_n + \bar{\mathbb{D}} \cdot \bar{V}_{\tau n} + \\ + \sum_{p=1}^{n} \bar{x}_1^{\,p}\, \bar{\mathbb{H}}_p : \mathbb{K}\, \bar{u}_{n-p} + \sum_{p=1}^{n} \bar{x}_1^{\,p} \left(\bar{\mathbb{H}}_p \cdot \bar{\mathbb{D}} \right) \cdot \bar{V}_{\tau n-p} = 0 . \tag{92}$$

The structure of these systems is of remarkable simplicity. In fact, let us pose

$$
\begin{aligned}
\mathcal{L}_{n-1} &= \int_0^{\bar{x}_1} \left\{ \frac{\partial \bar{\rho}_{n-1}}{\partial \bar{x}_0} + \bar{u}_{n-1} \bar{K} + \sum_{p=2}^{n} \bar{u}_{n-p}\, \bar{x}_1^{\,p}\, K_1 \mathbb{H}_p \right\} d\bar{x}_1 , \\
\mathbb{T}_{n-1} &= \int_0^{\bar{x}_1} \left\{ \frac{\partial \bar{\mathbb{V}}_{r_{n-1}}}{\partial \bar{x}_0} \cdot \mathbb{I}_r + \bar{\mathbb{D}} \bar{\rho}_{n-1} + \varpi\, \bar{\mathbb{D}} \bar{s}_{n-1} + \right. \\
&\qquad \left. + \sum_{p=1}^{n} \bar{x}_1^{\,p}\, \mathbb{H}_p \cdot \left(\bar{\mathbb{D}}\, \bar{\rho}_{n-p} + \varpi\, \mathbb{D} \bar{s}_{n-p} \right) \right\} d\bar{x}_1 ,
\end{aligned}
\tag{93}
$$

and let us note that these expressions can be calculated if the quantities of rank $p \le n - 1$ are known and continuously differentiable in x_0, a_1, a_2; this will lead to

$$
\left\{
\begin{aligned}
\bar{s}_n &= \bar{s}_{n-1} + \Sigma_n , \\
\bar{u}_n &= \mathcal{L}_{n-1} + U_n , \\
\bar{\rho}_n &= \mathcal{R}_n , \\
\bar{\mathbb{V}}_{r_n} &= \mathbb{T}_{n-1} + \mathbb{W}_n ,
\end{aligned}
\right.
\tag{94}
$$

agreeing that

$$
\bar{s}_{-1} \equiv \mathcal{L}_{-1} \equiv \mathbb{T}_{-1} \equiv 0 , \qquad u_{-1} \equiv \rho_{-1} \equiv s_{-1} \equiv \mathbb{V}_{r_{-1}} \equiv 0 ,
\tag{95}
$$

by means of which the system of rank n (including n = 0) is reduced to the following uncoupled system:

$$
\frac{\partial \Sigma_n}{\partial \bar{x}_1} = \frac{\partial\, U_n - \mathcal{R}_n}{\partial \bar{x}_1} = \frac{\partial \mathbb{W}_n}{\partial \bar{x}_1} = 0
\tag{96}
$$

/IV,51

$$
\frac{\partial}{\partial x_0}\left(U_n + \mathcal{R}_n + \varpi \Sigma_n \right) + \bar{K}\, U_n + \bar{\mathbb{D}} \cdot \mathbb{W}_n + \mathcal{M}_{n-1} = 0 ,
\tag{97}
$$

after having posed

$$
\mathcal{M}_{n-1} = \frac{\partial}{\partial x_0}\left(\mathcal{L}_{n-1} + \varpi\, \bar{s}_{n-1} \right) + \bar{K}\, \mathcal{L}_{n-1} + \mathbb{D} \cdot \mathbb{T}_{n-1}
\tag{98}
$$

$$+\sum_{p=1}^{n} \bar{x}_1^{\,p}\, \mathbb{H}_p : \bar{\mathbb{K}}\ \bar{u}_{n-p} + \sum_{p=1}^{n} \bar{x}_1^{\,p} (\bar{\mathbb{H}}_p \cdot \bar{\mathbb{D}}) \cdot \bar{V}_{T_{n-p}}$$

by noting that $\mathfrak{M}_{n-1}$ can be calculated if all quantities of rank $p \leq n - 1$ are known and if these are twice continuously differentiable with respect to x_0, a_1, a_2. We will perform here only formal operations and will assume that all required differentiability properties are fulfilled.

The system (96) shows that we have

$$\begin{cases} \Sigma_n = \Sigma_n(x_0; a_1, a_2), \\ R_n = U_n(x_0, x_1; a_1, a_2) + \mathcal{R}_n(x_0; a_1, a_2), \\ \mathbb{W}_n = \mathbb{W}_n(x_0; a_1, a_2), \end{cases} \tag{99}$$

while a substitution into eq.(97) will yield

$$\frac{\partial}{\partial x_0}\left(2U_n + \mathcal{R}_n + \varpi \Sigma_n\right) + \bar{K}\, U_n + \bar{\mathbb{D}} \cdot \mathbb{W}_n + \mathcal{M}_{n-1} = 0 . \tag{100}$$

However, this is all that one can hope to obtain from our investigation. Nevertheless, the consequences of eq.(100) can be further exploited. For this purpose, let us imagine that the function $\mathfrak{A}(x_0; a_1, a_2)$, introduced previously, verifies

$$\frac{1}{\mathcal{A}} \frac{\partial \mathcal{A}}{\partial \bar{x}_0} = \bar{K}, \tag{101}$$

which permits us to bring eq.(100) to the following form: /IV,52

$$\frac{\partial}{\partial x_0}\left(U_n \sqrt{\mathcal{A}}\right) + \frac{1}{2}\sqrt{\mathcal{A}} \left\{ \frac{\partial \mathcal{R}_n + \varpi \Sigma_n}{\partial x_0} + \bar{\mathbb{D}} \cdot \mathbb{W}_n + \mathcal{M}_{n-1} \right\} = 0 . \tag{102}$$

Theorem 13: Let us assume that a very regular acoustic field in a homogeneous atmosphere comprises a component in the form of acoustic wave trains spread about a wave surface of parametric representation

$$\mathbb{P}(x_0; a_1, a_2) = \mathbb{P}_0(a_1, a_2) + c_0\, x_0\, \mathfrak{n}(a_1, a_2) . \tag{103}$$

If this wave train, whose width is time-invariant, admits an asymptotic representation for large values of x_0, this latter can be obtained by the following procedure: Let us introduce the space-time representation

$$\left\{ \begin{array}{l} t = x_0 + t_0 \\ \mathcal{M} = \mathbb{P}_0(a_1, a_2) + (c_0 x_0 + x_1)\, n(a_1, a_2) \end{array} \right. \tag{104}$$

and let us define a function $\mathfrak{A}(x_0;\, a_1,\, a_2)$ by

$$\mathcal{A}(x_0;\, a_1, a_2) = \left| \left\{ \left(\frac{\partial \mathbb{P}_0}{\partial a_1} + c_0 x_0 \frac{\partial n}{\partial a_1} \right) \wedge \left(\frac{\partial \mathbb{P}_0}{\partial a_2} + c_0 x_0 \frac{\partial n}{\partial a_2} \right) \right\} \cdot n \right| . \tag{105}$$

Let

$$\left\{ \Sigma_n(x_0;\, a_1, a_2)\,,\ \mathcal{Q}_n(x_0;\, a_1, a_2)\,,\ \mathbb{W}_n(x_0;\, a_1, a_2) \right\} \tag{106}$$

be an infinite sequence of arbitrary functions that are independent of x_1 and which verify

$$n \cdot \mathbb{W}_n = 0 . \tag{107}$$

Let, on the other hand,

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$$\left\{ F_n(x_1;\, a_1, a_2) \right\} \tag{108}$$

be an infinite sequence of arbitrary functions that are independent of x_0, so that the asymptotic representation in question can be expressed by the formulas

$$\left\{ \begin{array}{l} \mathbb{V}_1 \cong n c_0 \displaystyle\sum_{p=0}^{\infty} \left\{ \frac{F_p(x_1;\, a_1, a_2)}{\sqrt{\mathcal{A}(x_0;\, a_1, a_2)}} + U_p(x_0;\, a_1, a_2) + \mathcal{L}_{p-1}(x_0, x_1, a_1, a_2) \right\} \\ \qquad + c_0 \displaystyle\sum_{p=0}^{\infty} \left\{ \mathbb{W}_p(x_0;\, a_1, a_2) + \mathbb{T}_{p-1}(x_0, x_1, a_1, a_2) \right\} \\ \rho_1 \cong \rho \displaystyle\sum_{p=0}^{\infty} \left\{ \frac{F_p(x_1;\, a_1, a_2)}{\sqrt{\mathcal{A}(x_0;\, a_1, a_2)}} + U_p(x_0;\, a_1, a_2) + \mathcal{Q}_p(x_0, a_1, a_2) \right\} \\ \dfrac{S_1}{c_v} \cong \displaystyle\sum_{p=0}^{\infty} \Sigma_p(x_0;\, a_1, a_2) \end{array} \right. \tag{109}$$

$$p_1 = \rho_0 c_0^2 \rho_1 + g_{s0} S_1 \qquad \varpi = \frac{C_V g_{s0}}{\rho_0 c_0^2}$$

and, noting that

$$\left\{\begin{aligned} \bar{u}_p &= F_p + U_p + \mathcal{L}_{p-1} \\ \bar{\mathbb{V}}_{T_p} &= \mathbb{W}_p + \mathbb{T}_{p-1} \\ \bar{\rho}_p &= F_p + U_p + R_p , \end{aligned}\right. \tag{110}$$

we obtain the recurrence formulas

$$F_{-1} = U_{-1} = \mathbb{W}_{-1} = \mathbb{T}_{-1} = R_{-1} = \mathcal{L}_{-1} = \bar{u}_{-1} = \bar{\mathbb{V}}_{T_{-1}} = \bar{\rho}_{-1} = 0 \tag{111}$$

and

$$\left\{\begin{aligned} &\frac{\partial U_p \sqrt{A}}{\partial x_0} + \frac{1}{2}\sqrt{A}\left\{\frac{\partial R_p + \varpi \Sigma_p}{\partial x_0} + \mathbb{D}\cdot\mathbb{W}_p + \mathcal{M}_{p-1}\right\} = 0 , \\ &\mathcal{L}_{p-1} = \int_0^{x_1}\left\{\frac{\partial \bar{\rho}_{p-1}}{\partial x_0} + \bar{u}_{p-1}\frac{\partial A}{A\,\partial x_0} + \sum_{q=2}^{p} u_{p-q}\, x_1^q\, \mathbb{K} : \mathbb{H}_q\right\} dx_1 , \\ &\mathbb{T}_{p-1} = \int_0^{x_1}\left\{\frac{\partial \bar{\mathbb{V}}_{T_{p-1}}}{\partial x_0}\cdot \mathbb{I}_T + \mathbb{D}\bar{\rho}_{p-1} + \varpi\, \mathbb{D}\Sigma_{p-1} + \right. \\ &\qquad\qquad \left. + \sum_{q=1}^{p} x_1^q\, \mathbb{H}_q \cdot \left(\mathbb{D}\bar{\rho}_{p-q} + \varpi\, \mathbb{D}\Sigma_{p-q}\right)\right\} dx_1 , \\ &\mathcal{M}_{p-1} = \frac{\partial}{\partial x_0}\left(\mathcal{L}_{p-1} + \varpi \Sigma_{p-1}\right) + \mathcal{L}_{p-1}\frac{1}{A}\frac{\partial A}{\partial x_0} + \mathbb{D}\cdot\mathbb{T}_{p-1} \\ &\qquad + \sum_{q=1}^{p} x_1^q\, \mathbb{H}_q : \mathbb{K}\, \bar{u}_{p-q} + \sum_{q=1}^{p} x_1^q \left(\mathbb{H}_q\cdot\mathbb{D}\right)\cdot \bar{\mathbb{V}}_{T_{p-q}} , \end{aligned}\right. \tag{112}$$

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where $\mathbb{D}$ denotes the component of the gradient in the plane normal to $\mathbf{n}$ on the surface $x_1 = 0$ and where the quantities $\mathbb{H}_p$ are obtained by writing

$$\mathbb{V} = \mathfrak{n} \frac{\partial}{\partial x_1} + \mathbb{D} + \sum_{p=1}^{\infty} x_1^{p} \mathbb{H}_p \cdot \mathbb{D} \quad . \tag{113}$$

This representation no longer is valid on approaching a zero of $\mathfrak{A}$. It is impossible to define the arbitrary functions (106) and (108) without investigating the correlation of the wave train with the remainder of the acoustic field.

Let us assume that the wave train is parallel to a front beyond which the perturbations vanish identically. In this case, we have

$$\lim_{x_1 \to +\infty} (F_0 + U_0) = \lim_{x_1 \to \infty} \mathbb{W}_0 = \lim_{x_1 \to \infty} R_0 = \lim_{x_1 \to \infty} \Sigma_0 \tag{114}$$

from which it follows that

$$\mathcal{R}_0 \equiv \Sigma_0 \equiv \mathbb{W}_0 \equiv 0 \, , \tag{115}$$

and, consequently,

$$\frac{\partial U_0 \mathcal{A}}{\partial x_0} = 0 \quad . \tag{116}$$

There is no difficulty in setting

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$$U_0 \equiv 0 \, , \tag{117}$$

where $F_0(x_1; a_1, a_2)$ will remain arbitrary under the condition that

$$\lim_{x_1 \to \infty} F_0(x_1; a_1, a_2) = 0 \quad . \tag{118}$$

This yields

$$\begin{cases} \bar{u}_0 = F_0(\bar{x}_1; a_1, a_2) \, , \\ \bar{\rho}_0 = \bar{\Gamma}_0(\bar{x}_1; a_1, a_2) \, , \\ \bar{V}_{T_0} = 0 \, , \\ \bar{s}_0 = 0 \, . \end{cases} \tag{119}$$

We assume F_0 to be indefinitely differentiable; in practice this is never the case but, in practical application, the expansions are always limited to a small number of terms. We will also assume that $\mathfrak{A}$ and $\mathbb{P}$, $\mathfrak{n}$ are indefinitely differ-

entiable in x_0, a_1, a_2. Under these conditions, we have

$$\left\{\begin{aligned} \mathcal{C}_0 &= \frac{1}{\mathcal{A}}\frac{\partial}{\partial x_0}\int_0^{x_1} F_0(x_1'; a_1, a_2)\, dx_1' = 0 \\ \mathcal{T}_0 &= \mathbb{D}\left\{\frac{1}{\mathcal{A}}\int_0^{x_1} F_0(x_1'; a_1; a_2)\, dx_1'\right\} = \mathcal{T}_0(x_0, x_1, a_1, a_2) \\ \mathcal{M}_0 &= \mathbb{D}\cdot\mathcal{T}_0 = \mathbb{D}\cdot\left\{\mathbb{D}\left[\frac{1}{\mathcal{A}}\int_0^{x_1} F_0(x_1'; a_1, a_2)\, dx_1'\right]\right\} = \mathcal{M}_0(x_0, x_1, a_1, a_2) \end{aligned}\right. \tag{120}$$

and it can be proved that

$$\lim_{x_1 \to \infty} \mathcal{T}_0 = \lim_{x_1 \to \infty} \mathcal{M}_0 = 0 . \tag{121}$$

It is now possible to resume, for the rank 1, the preceding reasoning and to conclude that the following is valid:

$$\mathcal{R}_1 \equiv \mathcal{Z}_1 \equiv \mathbb{W}_1 \equiv 0 \tag{121a}$$

and /IV,56

$$\lim_{x_1 \to \infty} \left(F_1 + U_1\right) = 0 \tag{122}$$

with

$$\frac{\partial U_1 \sqrt{\mathcal{A}}}{\partial x_0} + \frac{1}{2}\sqrt{\mathcal{A}}\,\mathcal{M}_0 = 0 , \tag{123}$$

which makes us set

$$U_1 = -\frac{1}{2\sqrt{\mathcal{A}}}\int_0^{x_0} \mathcal{M}_0 \sqrt{\mathcal{A}}\, dx_0 = U_1(x_0, x_1, a_1, a_2). \tag{124}$$

We find that

$$\lim_{x_1 \to \infty} U_1 = 0 , \tag{125}$$

such that $F_1\ (x_1, a_1, a_2)$ is arbitrary under the condition that

$$\lim_{x_1 \to \infty} F_1 (x_1, a_1, a_2) = 0 , \tag{126}$$

which we assume to be indefinitely differentiable. On reasoning by recurrence, we come to the conclusion that

$$\mathcal{R}_p \equiv \mathcal{E}_p \equiv \mathbb{W}_p \equiv 0 \tag{127}$$

and that $F_p(x_1;\ a_1, a_2)$ is arbitrary and indefinitely differentiable; this verifies

$$\lim_{x_1 \to \infty} F_p (x_1; a_1, a_2) = 0 . \tag{128}$$

Theorem 14: Under the conditions of theorem 13, if the acoustic wave train which is parallel to a front beyond which the complete acoustic field is identically one, eqs.(109) will be simplified to /IV,57

$$\left\{ \begin{aligned} V_1 &\cong c_0 n \sum_{p=0}^{\infty} \left\{ \frac{F_p(x_1; a_1, a_2)}{\sqrt{A(x_0; a_1, a_2)}} + U_p(x_0; a_1, a_2) + \mathcal{L}_{p-1}(x_0, x_1; a_1, a_2) \right\} \\ &\quad + c_0 \sum_{p=1}^{\infty} \mathcal{T}_{p-1}(x_0, x_1; a_1, a_2), \\ \rho_1 &\cong \rho_0 \sum_{p=0}^{\infty} \left\{ \frac{F_p(x_1; a_1, a_2)}{\sqrt{A(x_0; a_1, a_2)}} + U_p(x_0; a_1, a_2) \right\}, \\ S_1 &\cong 0, \\ p_1 &\cong c_0^2 \rho_1 . \end{aligned} \right. \tag{129}$$

The functions $F_p(x_1;\ a_1, a_2)$ are arbitrary under the provision that they are indefinitely differentiable and that they verify

$$\lim_{x_1 \to \infty} F_p(x_1; a_1, a_2) = 0 \quad . \tag{130}$$

This yields the following recurrence relations:

$$\left\{ \begin{aligned} U_p &= -\frac{1}{2\sqrt{\mathcal{A}}} \int_0^{x_0} \sqrt{\mathcal{A}}\, \mathcal{M}_{p-1}\, dx_0 \\ \mathcal{L}_{p-1} &= -\frac{1}{2}\sqrt{\mathcal{A}} \int_0^{x_1} \mathcal{M}_{p-2}\, dx_1 \\ &\quad + \sum_{q=2}^{p} \mathbb{K} : \mathbb{H}_q \int_0^{x_1} x_1^q \left(\frac{F_{p-q}}{\sqrt{\mathcal{A}}} + U_{p-q} + \mathcal{L}_{p-q-1} \right) dx_1 \\ \Pi_{p-1} &= \int_0^{x_1} dx_1 \left\{ \frac{\partial \Pi_{p-2}}{\partial x_0} \cdot I_0 + \mathbb{D}\left(\frac{F_{p-1}}{\sqrt{\mathcal{A}}} + U_{p-1} \right) + \sum_{q=1}^{p} x_1^q\, \mathbb{H}_{p-q} \cdot \mathbb{D}\left(\frac{F_{p-q}}{\sqrt{\mathcal{A}}} + U_{p-q} \right) \right\} \\ \mathcal{M}_{p-1} &= \frac{1}{\mathcal{A}} \frac{\partial}{\partial x_0} \left(\mathcal{A}\, \mathcal{L}_{p-1} \right) + \mathbb{D} \cdot \Pi_{p-1} + \\ &\quad + \sum_{q=1}^{p} \mathbb{H}_q : \mathbb{K}\; x_1^q \left(\frac{F_{p-q}}{\sqrt{\mathcal{A}}} + U_{p-q} \right) + \\ &\quad + \sum_{q=1}^{p} x_1^q \left(\mathbb{H}_q \cdot \mathbb{D} \right) \cdot \Pi_{p-q} \quad . \end{aligned} \right. \tag{131}$$

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We leave to the reader the task of proving that the proposed scheme is entirely explicit - with the arbitrariness being close to the sequence of F_n - such that, for calculating the quantities of rank p, only quantities of rank p - 1 will be necessary and that the quantities of rank -1 are ≡ 0. This leaves a very large arbitrariness (an infinite sequence of functions of three variables), a circumstance that cannot be helped since theorem 14 expresses the absolute maximum of information that can possibly be obtained by a purely local study.

Let us now return to the case in which the atmosphere is nonhomogeneous.

Here, the conditions are less simple since eq.(84) is not the only parameter to be taken into consideration. It is also necessary to compare ℓ with the nonhomogeneity scale of the atmosphere H, defined by the condition

$$\frac{H}{f_0}\left|\frac{\partial f_0}{a_0 \partial x_0}\right| = O(1), \tag{132}$$

if f_0 is any one of the quantities relative to the nonperturbed atmosphere, so that ℓ must be compared to H. It is obvious that the above theory cannot be simply extended unless we have

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$$\frac{\ell}{H} \ll 1 , \tag{133}$$

a condition which is readily obtainable in the terrestrial atmosphere if L^* is the width of a Mach wave train associated with supersonic flight. Conversely, $\frac{L}{H}$ may become important which indicates that the effects of nonhomogeneity have an important incidence. In principle, it is possible to construct formal expansions, analogous to eq.(88), under the hypothesis $\epsilon \to 0$ at fixed $\frac{H}{L}$. In fact, for the theory of "sonic boom" it will be sufficient to limit the calculation to the first approximation, obtained on replacing eq.(31) by

$$\left\{\begin{aligned} & \frac{\partial S_1}{\partial x_1} = 0 , \\ & \frac{\partial}{\partial x_1}\left(\rho_0 V_1 - c_0 \rho_1\right) = 0 , \\ & \frac{\partial W_1}{\partial x_1} = 0 , \\ & \rho_0 c_0 \frac{\partial V_1}{\partial x_0} + c_0^2 \frac{\partial \rho_1}{\partial x_0} + g_{S_0} \frac{\partial S_1}{\partial x_0} + \\ & \quad + V_1\left(\rho_0 c_0 \frac{\partial V_0}{\partial x_1} \cdot n + \rho_0 c_0^2 K + c_0^2 \frac{\partial \rho_0}{\partial x_1} + g_{S_0} \frac{\partial S_0}{\partial x_1}\right) - \\ & \quad - \frac{\rho_1}{\rho_0}\left(c_0^3 \frac{\partial \rho_0}{\partial x_1} + c_0 g_{S_0} \frac{\partial S_0}{\partial x_1}\right) + c_0^2 \rho_1 \nabla \cdot W_0 + c_0 \rho_1 \frac{\partial c_0^2}{\partial x_1} + \end{aligned}\right. \tag{134}$$

$$+ c_0 S_1 \frac{\partial g_{s_0}}{\partial x_1} + \mathbb{V}_1 \cdot \left\{ \rho_0 c_0 \left[Dc_0 + 2(D\mathbb{V}_0)\cdot \mathbf{n} \right] + c_0^2 D\rho_0 + g_{s_0} DS_0 \right\} = 0 .$$

Proposition 15: In a nonhomogeneous atmosphere, possibly with wind, let us consider an acoustic wave train of width ℓ. Let us assume that we have /IV,60

$$\ell \ll L , \qquad \ell \ll H, \tag{136}$$

where L is the distance covered by the train along the investigated sound ray since its formation, and where H is the nonhomogeneity scale of the atmosphere. Let us, in addition, use the representation (104) and let $\mathfrak{A}$ be the function defined by eq.(105). Under the hypotheses (136) and if the acoustic field is negligible outside of the wave train, the structure of this latter is described by the formulas

$$\begin{cases} \mathbb{V}_1 = \dfrac{\mathbf{n}(x_0; a_1, a_2)\; q(x_0; a_1, a_2) \sqrt{c_0(x_0; a_1, a_2)}}{\sqrt{\rho_0(x_0; a_1, a_2)\, \mathcal{A}(x_0; a_1, a_2)}} F(x_1; a_1, a_2), \\ \rho_1 = \dfrac{\rho_0(x_0; a_1, a_2)}{c_0(x_0; a_1, a_2)} \mathbf{n}(x_0; a_1, a_2)\cdot \mathbb{V}_1(x_0, x_1; a_1, a_2), \\ S_1 = 0, \\ p_1 = \rho_0 c_0^2 \rho_1, \end{cases} \tag{137}$$

where the function q verifies that

$$\frac{1}{q}\frac{\partial q}{\partial x_0} = - \mathbf{n}(\nabla \mathbb{V}_0)\cdot \mathbf{n} - \mathbf{n}\cdot \nabla c_0 . \tag{138}$$

The function $F(x_1; a_1, a_2)$ is arbitrary.

Let us pose $v_1 = \mathbf{n} \cdot \mathbb{V}_1$, so that the last equation in the system (134) is reduced to

$$2\rho_0 c_0 \frac{\partial V_1}{\partial x_0} + V_1 \left\{ \rho_0 c_0 \frac{\partial \mathbb{V}_0}{\partial x_1} \cdot n + \rho_0 c_0^2 K + c_0^2 \frac{\partial \rho_0}{\partial x_1} + g_{s_0} \frac{\partial s_0}{\partial x_1} - \right.$$
$$\left. - c_0^2 \frac{\partial \rho_0}{\partial x_1} - g_{s_0} \frac{\partial s_0}{\partial x_1} + \rho_0 c_0 \nabla \cdot \mathbb{V}_0 + \rho_0 \frac{\partial c_0^2}{\partial x_1} \right\} = 0 . \tag{139}$$

Taking the calculations which led to theorem 12 into consideration, this will furnish /IV,61

$$\sqrt{\frac{|\mathbf{p}|}{\rho_0 A}} \frac{\partial}{\partial x_0} \left(\sqrt{\frac{\rho_0 A}{|\mathbf{p}|}} V_1 \right) + \frac{1}{2} \left\{ n \cdot (\nabla \mathbb{V}_0) \cdot n + (\Gamma_0 - 1) \nabla \cdot \mathbb{V}_0 \right\} = 0 , \tag{140}$$

where it is sufficient to prove that

$$n \cdot (\nabla \mathbb{V}_0) \cdot n + (\Gamma_0 - 1) \nabla \cdot \mathbb{V}_0 = - \frac{1}{c_0 |\mathbf{p}|} \frac{\partial c_0 |\mathbf{p}|}{\partial x_0} , \tag{141}$$

so as to finally be able to prove eq.(137). However, by posing

$$u_0 = \mathbb{V}_0 \cdot n \quad , \quad w_0 = c_0 + u_0 , \tag{142}$$

we had obtained, in eq.(58),

$$\frac{\partial u_0}{\partial x_0} - \frac{u_0}{c_0} \frac{\partial c_0}{\partial x_0} - w_0 \left[n \cdot (\nabla \mathbb{V}_0) \cdot n + (\Gamma_0 - 1) \nabla \cdot \mathbb{V}_0 \right] = \frac{\partial \mathbb{V}_0}{\partial t} n + \frac{\partial c_0}{\partial t}$$
$$= \frac{1}{|\mathbf{p}|} \frac{\partial w_0 |\mathbf{p}|}{\partial x_0} , \tag{143}$$

from which eq.(141) results immediately. We noted that $q = |\mathbf{p}|$.

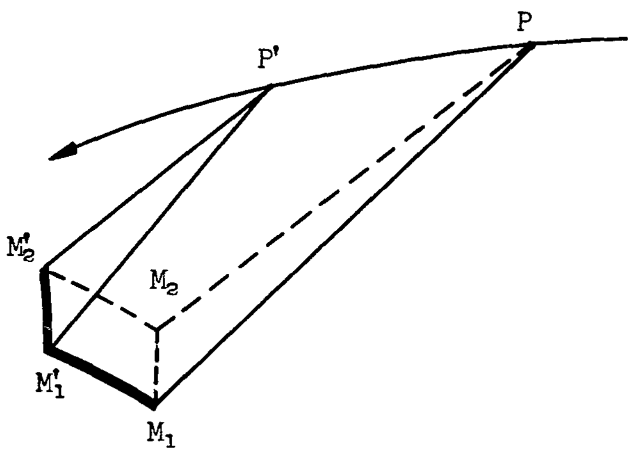
4.3 Application to the Mach Wave Train of a Supersonic Aircraft

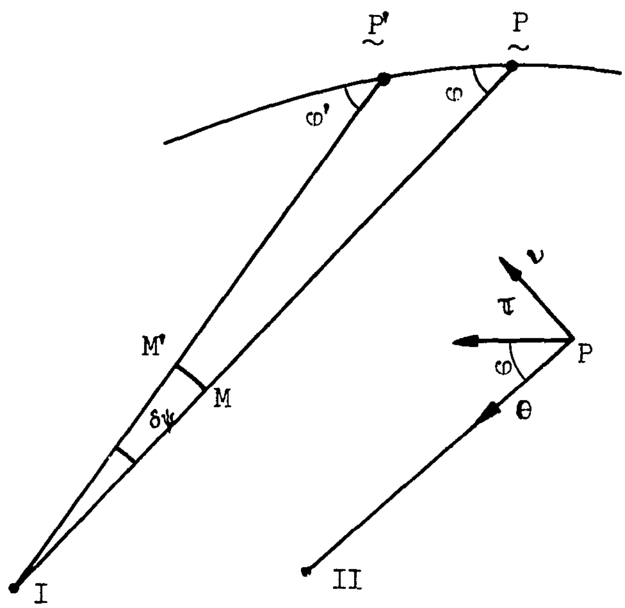
4.3.1 Homogeneous Atmosphere without Wind

In Section 1.4.5 we derived the proximal asymptotic behavior of the acoustic field of a supersonic aircraft (theorem 20); here, we will prove that this behavior is in agreement with the conclusions of theorem 14 of the present Chapter. For this, it is sufficient to demonstrate that

$$\frac{\rho_0 c_0^2 F_0(x_1; a_1, a_2)}{\sqrt{\mathcal{A}(x_0; a_1, a_2)}} = 2^{-1/2} \rho_0 V_0^2 \mathcal{M}_0^{\frac{1}{2}} \left\{a_0 T \mathcal{U}_0(T; \Theta)\right\}^{-\frac{1}{2}} G(x_1, \Theta). \quad (1)$$

Primarily, this leads to a definition of F_0 so that, basically, it remains to prove that $\mathfrak{A}$ varies with x_0 like $a_0 T \mathcal{U}_0(T, \Theta)$. It will be noted that, for /IV,62
a_1 and a_2, we can take the two parameters defining Θ, and that we can set $x_0 = T$.





It then remains to verify that the area of the curvilinear parallelogram $M_1 M_1' \Pi_2' \Pi_2$, traced by the Mach wave, varies like $a_0 T U_0 (T, \theta)$. Conversely, the side $M_1 \Pi_2$ varies proportionally to $a_0 T$, making it necessary to prove only that the side $M_1 M_1'$ varies like $U_0 (T, \theta)$. As shown in the accompanying diagram, MM' varies linearly with the distance MI from Π at the point of convergence I of the sound rays, so that MM' is linear in T, exactly like U_0. Thus, it is sufficient to demonstrate that $PI = a_0 T^*$ where T^* is given by eq.(91) of Chapter I, Sect.1.4. However,

$$PI = \lim_{P'P \to 0} \left(\frac{\delta \psi}{PP' \sin \varphi} \right)^{-1} = \left(\frac{\delta \psi}{V_a \, \delta t} \, \frac{1}{\sin \varphi} \right)^{-1} . \tag{2}$$

Let Θ be the unit vector carried by PI, let τ be the unit vector tangent in P to the trajectory, and let ν, which is unitary and orthogonal to Θ, be traced in the plane Θ, τ; this will yield

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$$d \cos \varphi = d \left(\frac{1}{\mu_a} \right) = \Theta \cdot d\tau + \tau \cdot d\Theta . \tag{3}$$

Let

$$\begin{aligned} \nu \cdot \tau \, d\psi &= - \frac{c}{V_a^2} \, dV_a - V_a \frac{n}{R} \cdot \Theta \, dt \\ &= - \left\{ \frac{c}{V_a^2} \gamma_\tau + \frac{1}{V_a} \gamma_n \cdot \Theta \right\} dt \\ &= - \frac{1}{V_a} \gamma \cdot \Theta \, dt \end{aligned} \tag{4}$$

and, since we have

$$\nu \cdot \tau = \sin \varphi = \frac{\sqrt{\mu_a^2 - 1}}{\mu_a} , \tag{5}$$

it follows that

$$PI = \frac{V_a^2 - c_0^2}{\gamma \cdot \Theta} . \tag{6}$$

This proves, in first approximation, the compatibility of the result formulated in theorem 20 of Chapter I with the result formulated in theorem 14 of the present Chapter.

4.3.2 Case of a Nonhomogeneous Atmosphere with Wind

Proposition 16: Let us characterize the Mach wave train produced by the flight of a supersonic aircraft, using the following parameters: $\mathfrak{R}$ is a generic variable derived from the two scalar variables specifying the characteristic sound ray; T is the transit time of the wave along this ray; x is the distance normal to the central wave of the train counted clockwise in the direction of propagation. Let $S_e(\xi;\mathfrak{R})$ be the law of areas of the equivalent fuselage /IV,64 for the direction of the ray $\mathfrak{R}$; using this function, let us form the Whitham function

$$F(\xi, R) = \frac{1}{2\pi}\int_{-\infty}^{\xi} \frac{S''_{e_{\xi\xi}}(\xi_1; R)}{\sqrt{\xi-\xi_1}}\, d\xi_1 . \tag{7}$$

The proximal asymptotic behavior - in the wave train - far from the aircraft, of the acoustic field of the latter is given by

$$\left\{\begin{array}{l} V = V_0 + c_0 V_1 n \\ p = p_0 + \rho_0 c_0 V_1 \\ \rho = \rho_0 + \rho_0 \dfrac{V_1}{c_0} \end{array}\right. \tag{8}$$

with

$$V_1 = V_a M_a^{3/2} (M_a^2 - 1) \left(2 c_{0a} T\right)^{-\frac{1}{2}} \left(\mathfrak{B}(T; R)\right)^{-1} F(-x M_a). \tag{9}$$

The vector n is the unit vector normal to the Mach wave; V_a, M_a, c_{0a} are, respectively, the flying speed, the Mach number of flight, and the sound speed at the instant of departure of the wave, at the point occupied by the aircraft at this instant. The function $\mathfrak{B}$ is given by

$$\mathfrak{B}(T; R) = \exp\left(\frac{1}{2}\int_0^T \left\{c_0 K - \frac{1}{T} + \frac{1}{\rho_0 c_0}\frac{\partial \rho_0 c_0}{\partial T} + (2\Omega - 1)\nabla V_0 + n(\nabla V_0)\cdot n\right\} dT\right). \tag{10}$$

The function $\mathfrak{B}$ is equal to unity if the atmosphere is homogeneous and without wind and if the flight is rectilinear and uniform. If the flight is accelerated in an isothermal atmosphere, we will have /IV,65

$$\mathcal{B} = \left(\frac{(\rho_0 c_0)_r}{(\rho_0 c_0)_a}\right)^{\frac{1}{2}} \left(1 - \frac{T}{T^*}\right)^{\frac{1}{2}} \qquad (7)$$

$$T^* \approx \frac{V_a^2 - c_{0a}^2}{c_0\, \gamma_R}$$

denoting by γ_R the projection of the acceleration vector onto the sound ray, counted clockwise in the direction of propagation. If the atmosphere is of arbitrary type, then $T^{-\frac{1}{2}}\mathcal{B}^{-1}$ will vary like the expression $\frac{q\sqrt{c_0}}{\sqrt{\rho_0}\,\mathfrak{A}}$ in proposition 15.

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CHAPTER V

/V.1

PHENOMENA OF DISSIPATION AND NONLINEAR CONVECTION IN PLANE WAVES AND IN GEOMETRIC ACOUSTICS

5.1 Euler Equations in Characteristic Variables

5.1.1 Preliminary Remarks

The selection of coordinates is the same as that given in Section 4.2.1; we will write the (exact) equations of the dynamics of gases (without viscosity and without heat conduction) in this system of coordinates. The notations are p, ρ, s, and - for the velocity vector -

$$\mathbb{V} = u_1 \, \mathbb{n} + u_2 \, \mathbb{e}_2 + u_3 \, \mathbb{e}_3 , \tag{1}$$

involving $\mathbf{V} = \mathbf{V}_0 + \mathbf{V}_1$, where $\mathbf{V}_0$ is the velocity vector of the wind and $\mathbf{V}_1$ is the velocity vector of the perturbation; for the time being, we will not linearize the equations. We have

$$\begin{aligned} \frac{D}{Dt} &= \frac{\partial}{\partial t} + \mathbb{V} \cdot \nabla \\ &= \frac{\partial}{\partial x_0} - c_0 \frac{\partial}{\partial x_1} + \left(u_1 - \mathbb{V}_0 \cdot \mathbb{n} \right) \frac{\partial}{\partial x_1} + \\ &\quad + \left\{ \left(u_2 \, \mathbb{e}_2 + u_3 \, \mathbb{e}_3 - \mathbb{V}_{0_T} \right) + x_1 \left[\mathbb{D} c_0 + (\mathbb{D} \mathbb{V}_0) \cdot \mathbb{n} \right] \right\} \cdot \mathbb{H} \cdot \mathbb{D} , \end{aligned}$$

which leads us to pose

$$\mathbb{V} - \mathbb{V}_0 = v_1 \, \mathbb{n} + v_2 \, \mathbb{e}_2 + v_3 \, \mathbb{e}_3 = v_1 \, \mathbb{n} + \mathbb{V}_T , \tag{2}$$

by means of which we obtain /V.2

$$\frac{D}{Dt} = \frac{\partial}{\partial x_0} + (v_1 - c_0) \frac{\partial}{\partial x_1} + \left(\mathbb{V}_T + x_1 \left[\mathbb{D} c_0 + (\mathbb{D} \mathbb{V}_0) \cdot \mathbb{n} \right] \right) \cdot \mathbb{H} \cdot \mathbb{D} . \tag{3}$$

Let us start by writing the continuity equation in explicit form; for this, we must write $(\mathbf{e}_1 = \mathbf{n})$ in explicit form

$$\nabla \cdot \mathbb{V} = \left(\mathbb{e}_1 \frac{\partial}{\partial x_1} + \mathbb{H} \cdot \mathbb{D} \right) \left(u_1 \mathbb{e}_1 + \mathbb{u}_T \right) \tag{4}$$
$$= \frac{\partial u_1}{\partial x_1} + u_1 \left(\mathbb{H} \cdot \mathbb{D} \right) \cdot \mathbb{e}_1 + \left(\mathbb{H} \cdot \mathbb{D} \right) \cdot \mathbb{u}_T ,$$

where we are using the notations

$$\mathbb{u}_T = u_2 \mathbb{e}_2 + u_3 \mathbb{e}_3 \quad . \tag{5}$$

Using the coordinates (x_2, x_3) of Section 4.2.1, we obtain $h_1 = H_1 + x_1 K_1$

$$\left(\mathbb{H} \cdot \mathbb{D} \right) \cdot \mathbb{e}_1 = \frac{H_2}{h_2} \mathbb{e}_2 \cdot \frac{1}{H_2} \frac{\partial \mathbb{e}_1}{\partial x_2} + \frac{H_3}{h_3} \mathbb{e}_3 \cdot \frac{1}{H_3} \frac{\partial}{\partial x_3} \mathbb{e}_1 \tag{6}$$
$$= \mathbb{H} : \mathbb{K} = \frac{H_2}{h_2} \frac{K_2}{H_2} + \frac{H_3}{h_3} \frac{K_3}{H_3} ,$$

since the coordinate curves x_2, x_3 are lines of curvature. Thus, the continuity equation is written as follows:

$$\frac{\partial \rho}{\partial x_0} + (V_1 - c_0) \frac{\partial \rho}{\partial x_1} + \left\{ \mathbb{V}_T + x_1 \left[\mathbb{D} c_0 + (\mathbb{D} \mathbb{V}_0) \cdot \mathbb{n} \right] \right\} \cdot \mathbb{H} \cdot \mathbb{D} \rho \tag{7}$$
$$+ \rho \left\{ \frac{\partial u_1}{\partial x_0} + u_1 \mathbb{H} : \mathbb{K} + \left(\mathbb{H} \cdot \mathbb{D} \right) \cdot \mathbb{u}_T \right\} = 0 \ .$$

Obviously, the entropy equation reads

$$\frac{\partial S}{\partial x_0} + (V_1 - c_0) \frac{\partial S}{\partial x_1} + \left(\mathbb{V}_T + x_1 \left[\mathbb{D} c_0 + (\mathbb{D} \mathbb{V}_0) \cdot \mathbb{n} \right] \right) \mathbb{H} \cdot \mathbb{D} S = 0 \tag{8}$$

which only leaves the equation of momentum to be written in explicit form. For this, we must write the following expression explicitly:

/V.3

$$\frac{D \mathbb{V}}{Dt} = \mathbb{e}_1 \left\{ \frac{\partial u_1}{\partial x_0} + (V_1 - c_0) \frac{\partial u_1}{\partial x_1} + \left(\mathbb{V}_T + x_1 \left[\mathbb{D} c_0 + (\mathbb{D} \mathbb{V}_0) \cdot \mathbb{n} \right] \right) \cdot \mathbb{H} \cdot \mathbb{D} u_1 \right\} \tag{9}$$
$$+ u_1 \left\{ \frac{\partial \mathbb{e}_1}{\partial x_0} + \left(\mathbb{V}_T + x_1 \left[\mathbb{D} c_0 + (\mathbb{D} \mathbb{V}_0) \cdot \mathbb{n} \right] \right) \cdot \mathbb{H} \cdot \mathbb{D} \mathbb{e}_1 \right\} +$$

$$+\left\{\frac{\partial \mathbb{U}_T}{\partial x_0} + (V_1 - C_0)\frac{\partial \mathbb{U}_T}{\partial x_1} + \left(\mathbb{V}_T + x_1\left[\mathbb{D}C_0 + (\mathbb{D}\mathbb{V}_0)\cdot \mathbf{n}\right]\right)\cdot \mathbb{H}\cdot \mathbb{D}\mathbb{U}_T\right\} = 0 .$$

In Section 4.2.2 we gave the formula material for facilitating the explicit writing of these equations, so that we will have, in particular,

$$\frac{\partial \mathbb{U}_T}{\partial x_0} = \frac{\partial \mathbb{U}_T}{\partial x_0}\cdot \mathbb{I}_T + \mathbf{n}\left\{\mathbb{U}_T\cdot \mathbb{D}C_0 + \mathbb{U}_T\cdot(\mathbb{D}\mathbb{V}_0)\cdot \mathbf{n}\right\} . \tag{10}$$

Let us consider the vector

$$\tau = \left\{\mathbb{V}_T + x_1\left[\mathbb{D}C_0 + (\mathbb{D}\mathbb{V}_0)\cdot \mathbf{n}\right]\right\}\cdot \mathbb{H} , \tag{11}$$

which obviously is located in the plane tangent to the wave; then, we must decompose $\tau \cdot \mathbb{D}\, \mathbf{e}_1$ and $\tau \cdot \mathbb{D}\, \mathbb{U}_T$ in accordance with $\mathbf{n}$ and in accordance with the plane tangent to the wave. Since we operate at x_0 = const, it is possible to use the coordinates x_2 and x_3, from which we obtain

$$\tau\cdot\mathbb{D} = \frac{\tau_2}{H_2}\frac{\partial}{\partial x_2} + \frac{\tau_3}{H_3}\frac{\partial}{\partial x_3} , \tag{12}$$

and, consequently,

$$\tau\cdot\mathbb{D}\,\mathbf{e}_1 = \frac{\tau_2 K_2}{H_2}\mathbf{e}_2 + \frac{\tau_3 K_3}{H_3}\mathbf{e}_3 = \mathbb{K}\cdot\tau . \tag{13}$$

Thus,

$$\begin{aligned}\tau\cdot\mathbb{D}\mathbb{U}_T &= (\tau\cdot\mathbb{D}u_2)\,\mathbf{e}_2 + (\tau\cdot\mathbb{D}u_3)\,\mathbf{e}_3 + \\ &\quad u_2\,\tau\cdot\mathbb{D}\,\mathbf{e}_2 + u_3\,\tau\cdot\mathbb{D}\,\mathbf{e}_3 \\ &= (\tau\cdot\mathbb{D}\,\mathbb{U}_T)\cdot\mathbb{I}_T - \tau\cdot\mathbb{K}\cdot\mathbb{U}_T\,\mathbf{e}_1\end{aligned} \tag{14}$$

if it is imagined that, since the coordinate lines are curvature lines, /V,4

$\frac{\partial \mathbf{e}_2}{\partial x_2}$, ... will have components in accordance with $\mathbf{e}_1$ that readily can be expressed by means of the quantities K_i. On assembling the results, we obtain

$$\frac{D\mathbb{V}}{Dt} = \mathbb{n}\left\{ \frac{\partial u_1}{\partial x_0} + (V_1 - c_0)\frac{\partial u_1}{\partial x_1} + \left(\mathbb{V}_T + x_1\left[\mathbb{D}c_0 + (\mathbb{D}\mathbb{V}_0)\cdot\mathbb{n}\right]\right)\cdot\mathbb{H}\cdot\mathbb{D}u_1\right\}$$

$$+ \mathbb{n}\left\{ \mathbb{u}_T\cdot\left[\mathbb{D}c_0 + (\mathbb{D}\mathbb{V}_0)\cdot\mathbb{n}\right] - \right.$$

$$\left. - \left(\mathbb{V}_T + x_1\left[\mathbb{D}c_0 + (\mathbb{D}\mathbb{V}_0)\cdot\mathbb{n}\right]\right)\cdot\mathbb{H}\cdot\mathbb{K}\cdot\mathbb{u}_T\right\} + \tag{15}$$

$$+ \mathbb{I}_T\cdot\frac{\partial\mathbb{u}_T}{\partial x_0} + (V_1 - c_0)\frac{\partial\mathbb{u}_T}{\partial x_1} - u_1\left(\mathbb{D}c_0 + (\mathbb{D}\mathbb{V}_0)\cdot\mathbb{n}\right) +$$

$$+ \left(\mathbb{V}_T + x_1\left[\mathbb{D}c_0 + (\mathbb{D}\mathbb{V}_0)\cdot\mathbb{n}\right]\right)\cdot\left\{(\mathbb{H}\cdot\mathbb{D}\,\mathbb{u}_T)\cdot\mathbb{I}_T + u_1\mathbb{H}\mathbb{K}\right\}.$$

<u>Theorem 1</u>: Let

$$\begin{cases} t = x_0 \\ \mathbb{x} = \mathbb{M} = \mathbb{P}(x_0, \mathfrak{R}) + x_1\,\mathbb{n}(x_0, \mathfrak{R}) \end{cases} \tag{16}$$

be a representation of instant points in a four-dimensional space-time continuum, where $\mathfrak{R}$ represents a system of two concealed undefined coordinates, such that $x_1 = 0$ represents an acoustic wave while $\mathfrak{R}$ remains invariant along the generating sound waves, involving an acoustic wave of the nonperturbed atmosphere. Let

$$\mathbb{V} - \mathbb{V}_0 = V_1\,\mathbb{n} + \mathbb{V}_T \tag{16a}$$

be the perturbation velocity vector and let

$$\mathbb{V} = u_1\,\mathbb{e}_1 + \mathbb{u}_T\,. \tag{17}$$

In addition, let $\mathbb{D}$ be the surface gradient vector, defined by the condition <u>/V,5</u> that, if f is a function of $\mathbb{P}$ on the surface $x_1 = 0$, $x_0 = \text{const}$, we will have

$$df = d\mathbb{P}\cdot\mathbb{D}f\,. \tag{18}$$

Again, let $\mathbb{I}_T$ be the unit tensor of vectorial space, tangent to the surface $x_1 = 0$, $x_0 = \text{const}$ and let $\mathbb{K}$ be the tensor of curvature of the same surface, defined by the condition that, for a displacement $d\,\mathbb{P}$ within the tangent plane, we have

$$d\,\mathbb{n} = d\mathbb{P}\cdot\mathbb{K}\,. \tag{19}$$

Finally, let H be the tangent vectorial space tensor, defined by

$$\mathbb{H}\cdot\left(\mathbb{I}_{\tau}+x_1\,\mathbb{K}\right)=\mathbb{I}_{\tau}\;. \tag{20}$$

Using this system of notations, together with the notations used until now, the equations of gas dynamics without dissipation can be written as follows:

Continuity:

$$\begin{aligned}\frac{\partial\rho}{\partial x_0}+(V_1-c_0)\frac{\partial\rho}{\partial x_1}+\left(\mathbb{V}_{\tau}+x_1\left[\mathbb{D}c_0+(\mathbb{D}\,\mathbb{V}_0)\cdot m\right]\right)\cdot\mathbb{H}\cdot\mathbb{D}\rho+\\ +\rho\left\{\frac{\partial u_1}{\partial x_1}+u_1\,\mathbb{H}:\mathbb{K}+(\mathbb{H}\cdot\mathbb{D}):\mathbb{u}_{\tau}\right\}=0\;.\end{aligned} \tag{21}$$

Entropy:

$$\frac{\partial s}{\partial x_0}+(V_1-c_0)\frac{\partial s}{\partial x_1}+\left\{\mathbb{V}_{\tau}+x_1\left[\mathbb{D}c_0+(\mathbb{D}\,\mathbb{V}_0)\cdot m\right]\right\}\cdot\mathbb{H}\cdot\mathbb{D}s=0\;. \tag{22}$$

Momentum: /V.6

$$\begin{aligned}\rho\Big\{\frac{\partial u_1}{\partial x_0}+(V_1-c_0)\frac{\partial u_1}{\partial x_1}+\left(\mathbb{V}_{\tau}+x_1\left[\mathbb{D}c_0+(\mathbb{D}\,\mathbb{V}_0)\cdot m\right]\right)\cdot\mathbb{H}\cdot\mathbb{D}u_1+\\ +\mathbb{u}_{\tau}\left(\mathbb{D}c_0+(\mathbb{D}\,\mathbb{V}_0)\cdot m\right)-\left(\mathbb{V}_{\tau}+x_1\left[\mathbb{D}c_0+(\mathbb{D}\,\mathbb{V}_0)\cdot m\right]\right)\cdot\mathbb{H}\cdot\mathbb{K}\cdot\mathbb{u}_{\tau}\Big\}\\ +\frac{\partial p}{\partial x_1}+\rho\,m\cdot\mathbb{g}=0\,,\end{aligned} \tag{23a}$$

$$\begin{aligned}\rho\Big\{\mathbb{I}_{\tau}\cdot\frac{\partial\mathbb{u}_{\tau}}{\partial x_0}+(V_1-c_0)\frac{\partial\mathbb{u}_{\tau}}{\partial x_1}-u_1\left(\mathbb{D}c_0+(\mathbb{D}\,\mathbb{V}_0)\cdot m\right)+\\ +\left(\mathbb{V}_{\tau}+x_1\left[\mathbb{D}c_0+(\mathbb{D}\,\mathbb{V}_0)\cdot m\right]\right)\cdot\left((\mathbb{H}\cdot\mathbb{D}\,\mathbb{u}_{\tau})\cdot\mathbb{I}_{\tau}+u_1\,\mathbb{H}\cdot\mathbb{K}\right)\Big\}+\\ +\mathbb{H}\cdot\mathbb{D}p+\rho\,\mathbb{I}_{\tau}\cdot\mathbb{g}=0\;.\end{aligned} \tag{23b}$$

Energy:

$$\left\{ \frac{\partial}{\partial x_0} + (V_1 - c_0)\frac{\partial}{\partial x_1} + \left(\mathbb{V}_T + x_1\left[\mathbb{D}c_0 + (\mathbb{D}\,V_0)\cdot \mathbb{n}\right]\right)\cdot \mathbb{H}\cdot\mathbb{D} \right\} \times$$

$$\times\left(\rho\left[e + \frac{u_1^2 + |\mathbb{u}_T|^2}{2}\right]\right) +$$

$$+ \frac{\partial}{\partial x_1}\left\{\rho\left(h + \frac{u_1^2 + |\mathbb{u}_T|^2}{2}\right) u_1\right\} + \tag{24}$$

$$+ \rho\left(h + \frac{u_1^2 + |\mathbb{u}_T|^2}{2}\right) u_1 \;\mathbb{H} : \mathbb{K} +$$

$$+ \mathbb{H}\cdot\mathbb{D}\cdot\left\{\rho\left(h + \frac{u_1^2 + |\mathbb{u}_T|^2}{2}\right)\mathbb{u}_T\right\} + \rho\,\mathbb{g}\cdot(u_1\mathbb{e}_1 + \mathbb{u}_T) = 0 .$$

Problem 1: Suppose that $\rho = \rho_0 + \rho_1$, $p = p_0 + p_1$, $S = S_0 + S_1$, $\mathbb{V} = \mathbb{V}_0 + \mathbb{V}_1$; linearize this and then return to eqs.(4.2.31).

5.1.2 Wave Trains of Small Width. Homogeneous Atmosphere /V,7

We assume a nonperturbed homogeneous atmosphere without wind and independent of time. Let us introduce the following dimensionless quantities:

$$\left\{\begin{array}{l} x_0 = \frac{L}{a_0}\bar{x}_0 \;,\quad x_1 = l\,\bar{x}_1 \;,\quad x_2 = L\,\bar{x}_2 \;,\quad x_3 = L\,\bar{x}_3 \\ \mathbb{K} = \frac{1}{L}\bar{\mathbb{K}} \;,\quad \mathbb{D} = \frac{1}{L}\bar{\mathbb{D}} \;,\quad \mathbb{H} = \mathbb{I}_T + \sum_{n=1}^{\infty}\left(\frac{l}{L}\right)^M \bar{x}_1^{\,n}\,\bar{\mathbb{H}}_m \\ p = p_0 + \rho_0 a_0^2\,\hat{p} \;,\quad \rho = \rho_0(1 + \hat{\rho}) \;,\quad S = S_0 + c_v\,\hat{S} \\ \mathbb{V} = a_0(\hat{V}_1\,\mathbb{n} + \hat{\mathbb{V}}_T) \\ \rho e = \rho_0 e_0 + \rho_0 c_0^2\,\widehat{\rho e} \;,\quad \rho h = \rho_0 h_0 + \rho_0 c_0^2\,\widehat{\rho h} \end{array}\right. \tag{25}$$

assuming that $\bar{x}_0$, $\bar{x}_1$, $\bar{x}_2$, $\bar{x}_3$, $\bar{K}$, $\bar{D}$, $\bar{H}_n$ and O(1) and that all $\hat{f}$ as well as $\frac{l}{L} \ll \ll 1$, and then let us finally attempt to exploit this situation. On substituting in the equations of motion, we obtain $(\bar{K} = \mathrm{trace}(\bar{\mathbb{K}}))$.

Continuity:

$$(\hat{V}_1-1)\frac{\partial\hat{\rho}}{\partial\bar{x}_1}+(1+\hat{\rho})\frac{\partial\hat{V}_1}{\partial\bar{x}_1}+\frac{\ell}{L}\Big\{\frac{\partial\hat{\rho}}{\partial\bar{x}_0}+\hat{V}_1(1+\hat{\rho})\bar{K}+$$
$$+\hat{\mathbb{V}}_T\cdot\bar{\mathbb{D}}\hat{\rho}+(1+\hat{\rho})\,\bar{\mathbb{D}}\circ\hat{\mathbb{V}}_T\Big\}+ \tag{26}$$
$$+\sum_{n=1}^{\infty}\left(\frac{\ell}{L}\right)^{n+1}\bar{x}_1^{\,n}\Big\{(1+\hat{\rho})\,\hat{V}_1\,\bar{\mathbb{H}}_n:\bar{\mathbb{K}}+(1+\hat{\rho})(\bar{\mathbb{H}}_n\cdot\bar{\mathbb{D}})\cdot\hat{\mathbb{V}}_T\;\boxed{+\hat{\mathbb{V}}_T\cdot\bar{\mathbb{H}}_n\cdot\bar{\mathbb{D}}\hat{\rho}}\Big\}=0$$

Entropy: /V,8

$$0=(\hat{V}_1-1)\frac{\partial\hat{s}}{\partial\bar{x}_1}+\frac{\ell}{L}\Big\{\frac{\partial\hat{s}}{\partial\bar{x}_0}+\hat{\mathbb{V}}_T\cdot\bar{\mathbb{D}}\hat{s}\Big\}+\sum_{n=1}^{\infty}\left(\frac{\ell}{L}\right)^{n+1}\bar{x}_1^{\,n}\,\hat{\mathbb{V}}_T\cdot\bar{\mathbb{H}}\cdot\mathbb{D}\hat{s} \tag{27}$$

Longitudinal momentum:

$$(1+\hat{\rho})(\hat{V}_1-1)\frac{\partial\hat{V}_1}{\partial\bar{x}_1}+\frac{\partial\hat{p}}{\partial\bar{x}_1}+\frac{\ell}{L}\Big\{(1+\hat{\rho})\Big[\frac{\partial\hat{V}_1}{\partial\bar{x}_0}+$$
$$+\hat{\mathbb{V}}_T\cdot\mathbb{D}\hat{V}_1-\hat{\mathbb{V}}_T\cdot\bar{\mathbb{K}}\cdot\hat{\mathbb{V}}_T+ \tag{28}$$
$$+\sum_{n=1}^{\infty}\left(\frac{\ell}{L}\right)^{n+1}\bar{x}_1^{\,n}(1+\hat{\rho})\Big\{-\hat{\mathbb{V}}_T\,\bar{\mathbb{H}}_n\cdot\bar{\mathbb{K}}\cdot\hat{\mathbb{V}}_T+\hat{\mathbb{V}}_T\cdot\bar{\mathbb{H}}_n\cdot\bar{\mathbb{D}}\hat{V}_1\Big\}=0$$

Lateral momentum:

$$(\hat{V}_1-1)(1+\hat{\rho})\frac{\partial\hat{\mathbb{V}}_T}{\partial\bar{x}_1}+\frac{\ell}{L}\Big\{\bar{\mathbb{D}}\hat{p}+\mathbb{I}_T\frac{\partial\hat{\mathbb{V}}_T}{\partial\bar{x}_0}(1+\hat{\rho})$$
$$+(1+\hat{\rho})\big(\hat{\mathbb{V}}_T\cdot(\bar{\mathbb{D}}\hat{\mathbb{V}}_T)\cdot\mathbb{I}_T+\hat{V}_1\,\hat{\mathbb{V}}_T\cdot\bar{\mathbb{K}}\big)\Big\}+ \tag{29}$$
$$+\sum_{n=1}^{\infty}\left(\frac{\ell}{L}\right)^{n+1}\bar{x}_1^{\,n}\Big\{\bar{\mathbb{H}}_n\cdot\bar{\mathbb{D}}\hat{p}+\big[\hat{\mathbb{V}}_T\cdot\bar{\mathbb{H}}_n\cdot(\bar{\mathbb{D}}\hat{\mathbb{V}}_T)\cdot\mathbb{I}_T+$$

$$+ \hat{v}_1 \; \hat{\mathbb{V}}_T \cdot \bar{\mathbb{H}}_M \cdot \bar{\mathbb{K}} \Big] (1 + \hat{\rho}) \Big\} = 0$$

Energy:

$$(\hat{v}_1 - 1) \frac{\partial}{\partial \bar{x}_1} \left\{ \widehat{\rho e} + (1+\hat{\rho}) \frac{\hat{v}_1^2 + |\hat{\mathbb{V}}_T|^2}{2} \right\} + \tag{30}$$

$$+ \frac{\partial}{\partial \bar{x}_1} \left\{ \left(\frac{h_0}{c_0^2} + \widehat{\rho h} + (1+\hat{\rho}) \frac{\hat{v}_1^2 + |\hat{\mathbb{V}}_T|^2}{2} \right) \hat{v}_1 \right\} +$$

/V,9

$$+ \frac{\ell}{L} \left\{ \frac{\partial}{\partial \bar{x}_0} \left(\widehat{\rho e} + (1+\hat{\rho}) \frac{\hat{v}_1^2 + |\hat{\mathbb{V}}_T|^2}{2} \right) + \right.$$

$$+ \hat{\mathbb{V}}_T \cdot \bar{\mathbb{D}} \left(\widehat{\rho e} + (1+\hat{\rho}) \frac{\hat{v}_1^2 + |\hat{\mathbb{V}}_T|^2}{2} \right) + \left(\frac{h_0}{c_0^2} + \widehat{\rho h} + (1+\hat{\rho}) \frac{\hat{v}_1^2 + |\hat{\mathbb{V}}_T|^2}{2} \right) \hat{v}_1 \bar{\mathbb{K}}$$

$$+ \bar{\mathbb{D}} \cdot \left[\left(\frac{h_0}{c_0^2} + \widehat{\rho h} + (1+\hat{\rho}) \frac{\hat{v}_1^2 + |\hat{\mathbb{V}}_T|^2}{2} \right) \hat{\mathbb{V}}_T \right] \Big\} +$$

$$+ \sum_{m=1}^{\infty} \left(\frac{\ell}{L} \right)^{m+1} \bar{x}_1^m \left\{ \hat{\mathbb{V}}_T \cdot \bar{\mathbb{H}}_m \cdot \bar{\mathbb{D}} \left(\widehat{\rho e} + (1+\hat{\rho}) \frac{\hat{v}_1^2 + |\hat{\mathbb{V}}_T|^2}{2} \right) + \right. \tag{31}$$

$$+ \left(\frac{h_0}{c_0^2} + \widehat{\rho h} + (1+\hat{\rho}) \frac{\hat{v}_1^2 + |\hat{\mathbb{V}}_T|^2}{2} \right) \hat{v}_1 \; \bar{\mathbb{H}}_m : \bar{\mathbb{K}} +$$

$$+ \bar{\mathbb{H}}_m \cdot \bar{\mathbb{D}} \left[\left(\frac{h_0}{c_0^2} + \widehat{\rho h} + (1+\hat{\rho}) \frac{\hat{v}_1^2 + |\hat{\mathbb{V}}_T|^2}{2} \right) \hat{\mathbb{V}}_T \right] \Big\} = 0 \, .$$

The preceding equations are exact and are simply written in a form convenient for studying the proximal asymptotic behavior of a nonlinear wave train if the width of the train is very small with respect to the distance covered by the train since its generation, which we interpret as

$$\varepsilon = \frac{\ell}{L} \ll 1 \, . \tag{32}$$

In acoustics, we know that $\hat{S} = 0$; $\hat{p}, \hat{\rho}, \hat{v}_1 \sim O(\epsilon)$; $V_T \sim O(\epsilon^2)$. Here, we do not know a priori the order of the perturbation and thus cannot afford to transpose without proper precautions the results derived in the acoustic investigation. Therefore, we will start the investigation by attempting to take advantage /V,10
of the single fact that ϵ and $\hat{f}$ are small, and nothing more. Thus, neglecting all terms that are small with respect to the already retained terms, we finally will retain, in first approximation,

$$\left\{\begin{array}{l} \dfrac{\partial\, \hat{v}_1 - \hat{\rho}}{\partial \bar{x}_1} = \dfrac{\partial \hat{S}_1}{\partial \bar{x}_1} = \dfrac{\partial\, \hat{p} - \hat{v}_1}{\partial \bar{x}_1} = 0, \\ \dfrac{\partial \hat{V}_T}{\partial \bar{x}_1} = 0, \\ \dfrac{p_0}{c_0^2} \dfrac{\partial \hat{v}_1}{\partial \bar{x}_1} - \dfrac{\partial \widehat{\rho e}}{\partial \bar{x}_1} = 0 . \end{array}\right. \tag{33}$$

This makes us believe that, again in first approximation, one of the results of geometric acoustics is conserved:

$$\left\{\begin{array}{l} \hat{p} \cong \hat{\rho} \cong \hat{v}_1 , \\ \hat{S} \cong 0 , \\ \hat{V}_T \cong 0 . \end{array}\right. \tag{34}$$

In addition, the energy equation yields

$$p_0 \hat{v}_1 \cong p_0 \hat{\rho} = c_0^2 \, \widehat{\rho e} , \tag{35}$$

which is a consequence of $\hat{p} \sim \hat{\rho} \sim \hat{v}_1$, $\hat{S} \sim 0$. This represents an already known phenomenon. Conversely, we still do not know the order of magnitude of the common value $\hat{p} \approx \hat{\rho} \approx \hat{v}_1$ relative to ϵ. To define this order, we will make use of eq.(34) and pose

$$\hat{p} = \hat{v}_1 (1 + \tilde{p}) , \quad \hat{\rho} = \hat{v}_1 (1 + \tilde{\rho}) , \quad \hat{S} = \hat{v}_1 \tilde{S} , \quad \hat{V}_T = \hat{v}_1 \tilde{V}_T \tag{36}$$

stipulating that $\tilde{p}$, $\tilde{\rho}$, $\tilde{S}$, $\tilde{V}_T$ are small but disregarding, for the moment, /V,11
the quantitative significance of this. We then substitute in the equations of motion and rearrange.

Continuity:

$$2\hat{V}_1\frac{\partial \hat{V}_1}{\partial \bar{x}_1}-\frac{\partial \hat{V}_1\tilde{\rho}}{\partial \bar{x}_1}+\frac{\partial}{\partial \bar{x}_1}\left(\hat{V}_1^2\tilde{\rho}\right)+$$
$$+\varepsilon\left\{\frac{\partial \hat{V}_1}{\partial \bar{x}_0}+\hat{V}_1\bar{K}+\frac{\partial \hat{V}_1\tilde{\rho}}{\partial \bar{x}_0}+(1+\tilde{\rho})\hat{V}_1^2\bar{K}+\right.$$
$$\left.+\bar{\mathbb{D}}\cdot(\hat{V}_1\tilde{\mathbb{V}}_T)+\bar{\mathbb{D}}\cdot\left[\hat{V}_1^2(1+\tilde{\rho})\tilde{\mathbb{V}}_T\right]\right\}+ \tag{37}$$
$$+\sum_{n=1}^{\infty}\varepsilon^{n+1}\bar{x}_1^{\,n}\left\{\hat{V}_1\bar{\mathbb{H}}_n:\bar{\mathbb{K}}+(1+\tilde{\rho})\hat{V}_1^2\bar{\mathbb{H}}_n:\bar{\mathbb{K}}+\right.$$
$$\left.+(\bar{\mathbb{H}}_n\cdot\bar{\mathbb{D}})\cdot(\tilde{\mathbb{V}}_T\hat{V}_1)+\bar{\mathbb{H}}_n\cdot\bar{\mathbb{D}}\cdot\left[(1+\tilde{\rho})\hat{V}_1^2\tilde{\mathbb{V}}_T\right]\right\}=0$$

<u>Entropy</u>:

$$-\frac{\partial \hat{V}_1\tilde{S}}{\partial \bar{x}_1}+\hat{V}_1\frac{\partial \hat{V}_1\tilde{S}}{\partial \bar{x}_1}+\varepsilon\left\{\frac{\partial \hat{V}_1\tilde{S}}{\partial \bar{x}_0}+\hat{V}_1\tilde{\mathbb{V}}_T\cdot\bar{\mathbb{D}}(\hat{V}_1\tilde{S})\right\}+$$
$$+\sum_{n=1}^{\infty}\varepsilon^{n+1}\bar{x}_1^{\,n}\hat{V}_1\tilde{\mathbb{V}}_T\cdot\mathbb{H}\cdot\bar{\mathbb{D}}(\hat{V}_1\tilde{S})=0 \tag{38}$$

<u>Longitudinal momentum</u>:

$$\frac{\partial \hat{V}_1\tilde{p}}{\partial \bar{x}_1}+\left[\hat{V}_1^2(1+\tilde{\rho})-\hat{V}_1\tilde{\rho}\right]\frac{\partial \hat{V}_1}{\partial \bar{x}_1}+$$
$$+\varepsilon\left\{\frac{\partial \hat{V}_1}{\partial \bar{x}_0}+\hat{V}_1(1+\tilde{\rho})\frac{\partial \hat{V}_1}{\partial \bar{x}_0}+\left(1+\hat{V}_1(1+\tilde{\rho})\right)\cdot\right.$$
$$\left.\cdot\left(\hat{V}_1\tilde{\mathbb{V}}_T\cdot\mathbb{D}\hat{V}_1-\hat{V}_1^2\tilde{\mathbb{V}}_T\cdot\mathbb{K}\cdot\tilde{\mathbb{V}}_T\right)\right\}+\sum_{n=1}^{\infty}\varepsilon^{n+1}\bar{x}_1^{\,n}\left(1+\hat{V}_1(1+\tilde{\rho})\right)\cdot \tag{39}$$
$$\left\{\hat{V}_1\tilde{\mathbb{V}}_T\cdot\bar{\mathbb{H}}_n\cdot\bar{\mathbb{D}}\,\hat{V}_1V-\hat{V}_1^2\tilde{\mathbb{V}}_T\cdot\mathbb{H}\cdot\mathbb{K}\cdot\tilde{\mathbb{V}}_T\right\}=0$$

Transverse momentum: /V,12

$$
\begin{aligned}
&(-1+\hat{v}_1+\hat{v}_1^2\tilde{\rho}-\hat{v}_1\tilde{\rho})\frac{\partial \hat{v}_1 \tilde{\mathbb{v}}_T}{\partial \bar{x}_1}+\varepsilon\Big\{\bar{\mathbb{D}}\hat{v}_1+\bar{\mathbb{D}}(\hat{v}_1\tilde{p})+\\
&\quad+[1+\hat{v}_1(1+\tilde{\rho})]\Big(\mathbb{I}_T\frac{\partial \hat{v}_1 \tilde{\mathbb{v}}_T}{\partial \bar{x}_0}+\hat{v}_1\tilde{\mathbb{v}}_T\cdot(\mathbb{D}(\hat{v}_1\tilde{\mathbb{v}}_T))\cdot\mathbb{I}_T+\hat{v}_1^2\tilde{\mathbb{v}}_T\cdot\bar{\mathbb{K}}\Big)\Big\}+\\
&+\sum_{n=1}^{\infty}\varepsilon^{n+1}\bar{x}_1^{\,n}\Big\{\bar{\mathbb{H}}_n\cdot\bar{\mathbb{D}}\hat{v}_1+\bar{\mathbb{H}}_n\mathbb{D}(\hat{v}_1\tilde{p})+\\
&+[1+\hat{v}_1(1+\tilde{\rho})]\Big(\hat{v}_1\tilde{\mathbb{v}}_T\cdot\bar{\mathbb{H}}_n\cdot(\bar{\mathbb{D}}(\hat{v}_1\tilde{\mathbb{v}}_T))\cdot\mathbb{I}_T+\hat{v}_1^2\tilde{\mathbb{v}}_T\cdot\bar{\mathbb{H}}_n:\bar{\mathbb{K}}\Big)\Big\}=0\,.
\end{aligned}
\tag{40}
$$

For writing the energy equation, it is necessary to estimate $\hat{\rho e}$, i.e.,

$$
\begin{aligned}
\rho_0 c_0^2\,\hat{\rho e} &= \rho e-\rho_0 e_0 = e_0(\rho-\rho_0)+(\rho_0+\rho-\rho_0)(e-e_0)\\
&= e_0(\rho-\rho_0)+(\rho_0+\rho-\rho_0)\Big\{\frac{p_0}{\rho_0^2}(\rho-\rho_0)+T_0(s-s_0)+\\
&\quad+\Big(\frac{c_0^2}{\rho_0^2}-\frac{2p_0}{\rho_0^3}\Big)\frac{1}{2}(\rho-\rho_0)^2\Big\}\,,
\end{aligned}
\tag{41}
$$

which induces us to pose

$$\hat{\rho e}=\frac{h_0}{c_0^2}\hat{v}_1(1+\tilde{\rho e}) \tag{42}$$

with

$$\tilde{\rho e}=\tilde{\rho}+\frac{1}{2}\frac{c_0^2}{h_0}\hat{v}_1+\frac{c_v T_0}{h_0}\tilde{S}+\dots\,. \tag{43}$$

In addition, we have

$$\hat{\rho h}=\hat{\rho e}+\hat{p} \tag{44}$$

such that we can write

$$\hat{\rho h}=\Big(\frac{h_0}{c_0^2}+1\Big)\hat{v}_1+\hat{v}_1\Big(\tilde{p}+\frac{h_0}{c_0^2}\tilde{\rho e}\Big)\,. \tag{45}$$

On substituting eq.(42) and (44) into the energy equation and limiting the calculation systematically to terms of the following type /V,13

cubic in $\tilde{f}$, $\hat{v}_1$

ϵx quadratic in $\tilde{f}$, $\hat{v}_1$
ϵ^2x linear in $\tilde{f}$, $\hat{v}_1$

where this choice is dictated by the fact that, as will be demonstrated below, $\tilde{p}$, $\tilde{\rho}$, $\tilde{V}_T$, $\hat{V}_1$ are $O(\epsilon)$ and by the fact that we are not using equations of an order higher than $O(\epsilon^3)$, we will obtain

$$\left(3\frac{h_0}{c_0^2}+1\right)\hat{v}_1\frac{\partial\hat{v}_1}{\partial\bar{x}_1}-\frac{h_0}{c_0^2}\frac{\partial\hat{v}_1\tilde{\rho e}}{\partial\bar{x}_1}+$$
$$+\hat{v}_1\left(2\tilde{p}+3\frac{h_0}{c_0^2}\tilde{\rho e}+\hat{v}_1\right)\frac{\partial\hat{v}_1}{\partial\bar{x}_1}+\hat{v}_1^2\frac{\partial}{\partial\bar{x}_1}\left(\tilde{p}+2\frac{h_0}{c_0^2}\tilde{\rho e}\right)$$
$$+\epsilon\left\{\!\!\left\{\frac{h_0}{c_0^2}\left(\frac{\partial\hat{v}_1}{\partial\bar{x}_0}+\hat{v}_1K\right)+\right.\right. \tag{46}$$
$$+\hat{v}_1\frac{\partial\hat{v}_1}{\partial\bar{x}_0}+\frac{h_0}{c_0^2}\frac{\partial\hat{v}_1\tilde{\rho e}}{\partial\bar{x}_0}+\left(\frac{h_0}{c_0^2}+1\right)\hat{v}_1^2K+$$
$$\left.\left.+\frac{h_0}{c_0^2}\bar{D}\cdot\left(\hat{v}_1\tilde{V}_T\right)\right\}\!\!\right\}+$$
$$+\epsilon^2\bar{x}_1\frac{h_0}{c_0^2}\hat{v}_1\,\mathbb{H}_1:\mathbb{K}+\cdots=0\,.$$

The above equations must be supplemented by the equations of state of the gas. Thus,

/V,14

$$\hat{v}_1(1+\tilde{p})=\frac{g\left(\rho_0[1+\hat{v}_1(1+\tilde{\rho})],\,s_0+c_v\hat{v}_1\tilde{s}\right)-g(\rho_0,s_0)}{\rho_0c_0^2} \tag{47}$$

permits to express $\tilde{p}$ as a function of the principal variables $\hat{v}_1$, $\tilde{\rho}$, $\tilde{S}$. Writing this in explicit form and taking into consideration $\frac{\partial g}{\partial\rho}=c^2$, we obtain

$$\tilde{p}=\tilde{\rho}+\frac{c_v}{\rho_0c_0^2}\frac{\partial g}{\partial s_0}\tilde{s}+\frac{\hat{v}_1}{2}\left\{\frac{\rho_0}{c_0^2}\frac{\partial^2 g}{\partial\rho_0^2}(1+\tilde{\rho})^2+\right. \tag{48}$$
$$\left.+\frac{c_v^2}{\rho_0c_0^2}\frac{\partial^2 g}{\partial s_0^2}(\tilde{s})^2+2\frac{\rho_0c_v}{c_0^2}\frac{\partial^2 g}{\partial s_0\partial\rho_0}\tilde{S}(1+\tilde{\rho})\right\}+\cdots,$$

i.e., in first approximation,

$$\tilde{p} = (\Gamma_0 - 1)\,\hat{v}_1 + \tilde{\rho} + \frac{c_v}{\rho_0 c_0^2}\,\frac{\partial g}{\partial S_0}\,\tilde{S} + \dots, \tag{49}$$

which thus is the analog of eq.(45).

Let us now return to the energy equation and to the other equations of motion, agreeing to disregard there any term that is definitely small with respect to any retained term, basing our calculation on the criterion that $\hat{v}_1$, $\tilde{\rho}$, $\tilde{p}$, $\tilde{s}$, $\tilde{V}_T$ are small and that $\bar{K}$, $\mathbb{D}$, $\frac{\partial}{\partial \bar{x}_1}$, ... are of the order of unity. This will yield:

Continuity:

$$2\,\hat{V}_1\,\frac{\partial \hat{V}_1}{\partial \bar{x}_1} - \frac{\partial \hat{V}_1 \tilde{\rho}}{\partial \bar{x}_1} + \varepsilon\left(\frac{\partial \hat{V}_1}{\partial \bar{x}_0} + \hat{V}_1 \bar{K}\right) = 0 \tag{50}$$

Longitudinal momentum [1]: /V,15

$$\frac{\partial \hat{V}_1 \tilde{p}}{\partial \bar{x}_1} + \varepsilon\,\frac{\partial \hat{V}_1}{\partial \bar{x}_0} = 0 \tag{51}$$

Energy[1]:

$$\left(3\,\frac{h_0}{c_0^2} + 1\right)\hat{V}_1\,\frac{\partial \hat{V}_1}{\partial \bar{x}_1} - \frac{h_0}{c_0^2}\,\frac{\partial \hat{V}_1 \widetilde{\rho e}}{\partial \bar{x}_1} + \frac{h_0}{c_0^2}\,\varepsilon\left(\frac{\partial \hat{V}_1}{\partial \bar{x}_0} + \hat{V}_1 \bar{K}\right) = 0 \tag{52}$$

Transverse momentum [1]:

$$\frac{\partial \hat{V}_1 \tilde{V}_T}{\partial \bar{x}_1} - \varepsilon\,\mathbb{D}\,\hat{V}_1 = 0 \tag{53}$$

Entropy [1] (outside of the shocks):

$$\frac{\partial \hat{V}_1 \tilde{S}}{\partial \bar{x}_1} = \varepsilon\,\frac{\partial V_1 \tilde{S}}{\partial \bar{x}_0}\,. \tag{54}$$

These equations indicate that $\hat{V}_1 = O(\epsilon)$ and that $\tilde{p} = O(\epsilon)$. So far as $\tilde{S}$ is

concerned, it is obvious that its variation at the order ϵ is zero outside of the shocks. However, a shock obviously has an intensity $O(\epsilon)$ so that $\tilde{S}$ will be $O(\epsilon^2)$ in such a manner that $\hat{V}_1\tilde{S} = O(\epsilon^3)$. Thus, it is possible to search for a systematic expansion of the following form:

$$\left\{\begin{aligned} \hat{V}_1 &= \varepsilon \tilde{V}_0 + \cdots \\ \tilde{p} &= \varepsilon \tilde{p}_0 + \cdots \\ \tilde{\rho} &= \varepsilon \tilde{\rho}_0 + \cdots \\ \tilde{S} &= \varepsilon^2 \tilde{S}_0 + \cdots \\ \tilde{W}_r &= \varepsilon \tilde{W}_{r_0} + \cdots \end{aligned}\right. \tag{55}$$

By substitution, we then find the system

/V,16

$$\left\{\begin{aligned} & 2\tilde{V}_0 \frac{\partial \tilde{V}_0}{\partial \bar{x}_1} - \frac{\partial \tilde{V}_0 \tilde{\rho}_0}{\partial \bar{x}_1} + \frac{\partial \tilde{V}_0}{\partial \bar{x}_0} + \tilde{V}_0 \bar{K} = 0, && \text{a)} \\ & \frac{\partial \tilde{V}_0 \tilde{p}_0}{\partial \bar{x}_1} + \frac{\partial \tilde{V}_0}{\partial \bar{x}_0} = 0, && \text{b)} \\ & \left(\frac{3h_0}{c_0^2} + 1\right) \tilde{V}_0 \frac{\partial \tilde{V}_0}{\partial \bar{x}_1} - \frac{h_0}{c_0^2} \frac{\partial}{\partial \bar{x}_1}\left(\tilde{V}_0 \left[\tilde{\rho}_0 + \frac{p_0}{\rho_0 h_0} \tilde{V}_0\right]\right) + \\ & \qquad + \frac{h_0}{c_0^2}\left(\frac{\partial \tilde{V}_0}{\partial \bar{x}_0} + \tilde{V}_0 \bar{K}\right) = 0, && \text{c)} \\ & \frac{\partial \tilde{V}_0 \tilde{W}_{r_0}}{\partial \bar{x}_1} - \bar{D}\, \tilde{V}_0 = 0, && \text{d)} \\ & \frac{\partial \tilde{V}_0 \tilde{S}_0}{\partial \bar{x}_1} = 0. \qquad \text{(outside of the shocks)} && \text{e)} \end{aligned}\right. \tag{56}$$

It should be noted that the resultant system is nonlinear. The wave trains which we are attempting to describe will have a structure differing basically from that of the acoustic wave train which we had studied in Chapter IV. Let us note that eq.(56) must be supplemented by

$$\tilde{p}_0 = (\Gamma_0 - 1)\, \tilde{V}_0 + \tilde{\rho}_0 \tag{57}$$

and let us also note that

$$\begin{cases} (56.a) & \longleftrightarrow & \text{continuity,} \\ (56.b) & \longleftrightarrow & \text{longitudinal momentum,} \\ (56.c) & \longleftrightarrow & \text{energy,} \\ (56.d) & \longleftrightarrow & \text{transverse momentum.} \end{cases} \qquad (58)$$

By adding eqs.(56a) and (56b) term by term and taking eq.(57) into consideration, we find

/V,17

$$2\Gamma_0 \tilde{V}_0 \frac{\partial \tilde{V}_0}{\partial \bar{x}_1} + 2\frac{\partial \tilde{V}_0}{\partial \bar{x}_0} + \tilde{V}_0 \bar{K} = 0 , \qquad (\Sigma_0, a) \qquad (59)$$

i.e., an equation that directly governs $\tilde{V}_0$. We then have three uncoupled equations for $\tilde{p}_0$, $\tilde{S}_0$, $\tilde{V}_{T0}$, i.e.,

$$\begin{cases} \dfrac{\partial \tilde{V}_0 \tilde{p}_0}{\partial \bar{x}_1} + \dfrac{\partial \tilde{V}_0}{\partial \bar{x}_0} = 0 , \\ \dfrac{\partial \tilde{V}_0 \tilde{S}_0}{\partial \bar{x}_1} = 0 , \quad \text{(outside of the shocks)} \; (\Sigma_0 , b) \\ \dfrac{\partial \tilde{V}_0 \tilde{V}_{T0}}{\partial \bar{x}_1} - \bar{D} \tilde{V}_0 = 0 , \end{cases} \qquad (60)$$

yielding finally

$$\tilde{f}_0 = \tilde{p}_0 - (\Gamma_0 - 1)\, \tilde{V}_0 . \qquad (\Sigma_0, c) \qquad (61)$$

The reader will be able to see that the energy equation, at the approximation order considered here, does not differ from the equation of continuity.

In a formal way, one can attempt to complete eq.(55) by an asymptotic expansion in power series of ϵ. On substituting in the equations of motion and canceling the successive power coefficients of ϵ, we obtain a hierarchy of differential systems of rank 1, 2, ... which are linear. In fact, there is nothing to guarantee that the asymptotic expansion of the solution proceeds in accordance with the powers. It is only possible to construct this series step by step, for which the connectivity with an asymptotic solution, valid outside of the wave train, in fundamental. /V,18

5.1.3 Shock Waves

Let us pick up the notations of theorem 1 and let us assume that a shock is located in $x_1 = x_{1_c}(x_0, \mathfrak{R})$ and then attempt to write the conditions of shock. First, it is necessary to define the unit vector $\mathbb{N}$ normal to the surface of this latter. Let us then proceed over the intermediary of the coordinates x_2 and x_3. Posing $h_2 = H_2 + x_{1_c}K_2$, $h_3 = H_3 + x_{1_c}K_3$, the vector $\mathbb{N}$ will become proportional to

$$\begin{aligned}\left(h_2 \mathbb{E}_2 + \frac{\partial x_{1c}}{\partial x_2} \mathbb{n}\right) \wedge \left(h_3 \mathbb{E}_3 + \frac{\partial x_{1c}}{\partial x_3} \mathbb{n}\right) &= h_2 h_3 \mathbb{n} + h_2 \frac{\partial x_{1c}}{\partial x_3} \mathbb{E}_2 \wedge \mathbb{n} + h_3 \frac{\partial x_{1c}}{\partial x_2} \mathbb{n} \wedge \mathbb{E}_3 \\ &= h_2 h_3 \left\{ \mathbb{n} - \mathbb{H} \cdot \mathbb{D}\, x_{1c} \right\},\end{aligned} \tag{62}$$

whence

$$\mathbb{N} = \frac{\mathbb{n} - \mathbb{H} \cdot \mathbb{D}\, x_{1c}}{\sqrt{1 + \|\mathbb{H} \cdot \mathbb{D}\, x_{1c}\|^2}}. \tag{63}$$

After this, it is necessary to determine the rate of normal displacement of the shock which we denote by W_c and which results from

$$W_c = \mathbb{N} \cdot \left(\frac{\partial \mathbb{P}}{\partial x_0} + \frac{\partial x_{1c}}{\partial x_0} \mathbb{n} + x_{1c} \frac{\partial \mathbb{n}}{\partial x_0} \right), \tag{64}$$

i.e.,

$$W_c = \frac{\mathbb{n} \cdot \mathbb{V}_0 + c_0 - \mathbb{V}_0 \cdot \mathbb{H} \cdot \mathbb{D}\, x_{1c}}{\sqrt{1 + \|\mathbb{H} \cdot \mathbb{D}\, x_{1c}\|^2}} + \frac{\frac{\partial x_{1c}}{\partial x_0}}{\sqrt{1 + \|\mathbb{H} \cdot \mathbb{D}\, x_{1c}\|^2}} + \frac{x_{1c} \left(\mathbb{D} c_0 + (\mathbb{D} \mathbb{V}_0) \cdot \mathbb{n} \right) \cdot \mathbb{H} \cdot \mathbb{D}\, x_{1c}}{\sqrt{1 + \|\mathbb{H} \cdot \mathbb{D}\, x_{1c}\|^2}} \tag{65}$$

With the conventional notation $f_+ - f_- = [f]$, the shock conditions /V,19 will be written as follows:

$$\left[\rho (\mathbb{V} - W_c \mathbb{N}) \cdot \mathbb{N} \right] = 0, \tag{66}$$

$$\left\{\begin{aligned} &\left[\, p\,\mathbb{N} + \rho(\mathbb{V} - w_c\mathbb{N})\,\mathbb{V} \right] = 0\,, \\ &\left[\, h + |\mathbb{V} - w_c\mathbb{N}|^2/2 \right] = 0\,, \end{aligned}\right.$$

of which we will write the expansion in explicit form, in its very first terms, assuming a homogeneous atmosphere and a small ϵ. In this case, we can write

$$\left\{\begin{aligned} &\mathbb{N} = \sum_{n=0}^{\infty} \varepsilon^n \bar{\mathbb{N}}_n \\ &W_c = c_0 \sum_{n=0}^{\infty} \varepsilon^n \bar{W}_{c_n} \end{aligned}\right. \tag{67}$$

with

$$\left\{\begin{aligned} &\bar{\mathbb{N}}_0 = n \\ &\bar{\mathbb{N}}_1 = -\bar{\mathbb{D}}\,\bar{x}_{1c} \\ &\bar{\mathbb{N}}_2 = -\tfrac{1}{2}|\bar{\mathbb{D}}\bar{x}_{1c}|^2\, n - \bar{\mathbb{H}}_1 \cdot \bar{\mathbb{D}}\,\bar{x}_{1c} \\ &\ldots \end{aligned}\right. \tag{68}$$

$$\left\{\begin{aligned} &\bar{W}_{c_0} = 1 \\ &\bar{W}_{c_1} = \frac{\partial \bar{x}_{1c}}{\partial \bar{x}_0}\,, \\ &\bar{W}_{c_2} = -\tfrac{1}{2}|\bar{\mathbb{D}}\bar{x}_{1c}|^2 \\ &\ldots \end{aligned}\right. \tag{69}$$

In addition, so far as W_c is concerned, no true expansion in series is involved here, since $\bar{x}_{1_c}$ may also depend on ϵ. Therefore, let us limit the calculation to the first terms:

/V,20

$$\Big[\big(1 + \hat{V}_1(1+\gamma)\big)\Big(\hat{V}_1\big(1 + \varepsilon^2\mathbb{N}_2\cdot n + \cdots\big) + \varepsilon\,\hat{V}_1\,\tilde{\mathbb{V}}_T\cdot\bar{\mathbb{N}}_1 - 1 - \varepsilon\,\bar{W}_c - \varepsilon^2\,\bar{W}_{c_2} + \cdots\Big)\Big] = 0\,, \tag{70}$$

$$\Big[\hat{V}_1(1+\tilde{p})(m+\varepsilon\bar{N}_1+\varepsilon^2 N_2)+\big(1+\hat{V}_1(1+\tilde{\rho})\big)\big(\hat{V}_1[1+\varepsilon^2\bar{N}_2\cdot m+\dots]$$

$$+\varepsilon\hat{V}_1\tilde{V}_T\cdot N_1\cdot-1-\varepsilon\bar{w}_{c_1}-\varepsilon^2\bar{w}_{c_2}\big)\big(\hat{V}_1 m+\hat{V}_1\tilde{V}_T\big)\Big]=0\,,$$

$$\Big[\Big(\frac{h_0}{c_0^2}+\Big(\frac{h_0}{c_0^2}+1\Big)\hat{V}_1+\hat{V}_1\Big(\tilde{p}+\frac{h_0}{c_0^2}\rho\tilde{e}\Big)\Big)\Big(1+\hat{V}_1(1+\tilde{\rho})\Big)^{-1}+$$

$$+\frac{1}{2}\Big|\Big(\hat{V}_1 m+\hat{V}_1\tilde{V}_T-n-\varepsilon\bar{w}_{c_1}m\underset{(-\varepsilon N_1)}{-}\varepsilon^2(\bar{w}_{c_2}m+\bar{w}_{c_1}\bar{N})\Big)\Big|^2\Big]=0.$$

Under these shock conditions, let us agree to retain only the terms that are not necessarily negligible with respect to the already retained terms; this will yield

$$\left\{\begin{array}{ll}\Big[\hat{V}_1\big(\varepsilon\bar{w}_{c_1}+\tilde{\rho}\hat{V}_1\big)\Big]=0\,, & a)\\ \Big[\hat{V}_1\{(\tilde{p}-\varepsilon\bar{w}_{c_1})m+\varepsilon(\bar{N}_1-\tilde{V}_T)\}\Big]=0\,, & b)\\ \Big[\hat{V}_1\Big(\tilde{p}+\frac{h_0}{c_0^2}\rho\tilde{e}-\frac{h_0}{c_0^2}\tilde{\rho}-\frac{\hat{V}_1}{2}-\varepsilon\bar{w}_{c_1}\Big)\Big]=0\,, & c)\end{array}\right. \tag{71}$$

to which, naturally, the following must be joined:

$$\left\{\begin{array}{ll}\rho\tilde{e}=\tilde{\rho}+\frac{1}{2}\frac{c_0^2}{h_0}\hat{V}_1+\frac{c_v\Gamma_0}{h_0}\tilde{S}+\dots\,, & a)\\ \tilde{p}=(\Gamma_0-1)\hat{V}_1+\tilde{\rho}+\frac{c_v}{\rho_0 c_0^2}\frac{\partial g}{\partial S_0}\tilde{S}+\dots\,. & b)\end{array}\right. \tag{72}$$

A combination of eqs.(71) and (72) will yield /V,21

$$\Big[\hat{V}_1\big(\varepsilon\bar{w}_{c_1}+\tilde{\rho}\big)\Big]=0\,, \qquad a) \tag{73}$$

$$
\left\{
\begin{array}{lr}
\left[\hat{V}_1\left(\Gamma_0 \hat{V}_1 - 2\varepsilon \bar{w}_{c_1}\right)\right] = 0 , & b) \\
\left[\hat{V}_1\left(\bar{\mathbb{N}}_1 - \tilde{V}_{\P}\right)\right] = 0 , & c) \\
\left[\hat{V}_1 \tilde{s}\right] = 0 . & d)
\end{array}
\right.
$$

Obviously, this latter relation is a priori expected, since it expresses the conservation of entropy on passage of the shock, which takes place in second approximation. Using the notations (55), we are induced to pose

$$
\bar{x}_{1c} = \bar{x}_{1c_0} + \cdots \tag{74}
$$

where $\bar{x}_{1c_0}$, this time, is independent of ϵ. We find that the system of the shock conditions is uncoupled like the system Σ_0, i.e., on the one hand

$$
\left[\tilde{V}_0\left(\Gamma_0 \tilde{V}_0 - 2 \frac{\partial \bar{x}_{1c_0}}{\partial \bar{x}_0}\right)\right] = 0 , \tag{75}
$$

which, at (Σ_0, a), must ensure the determination of $\tilde{V}_0(\bar{x}_0, \bar{x}_1, \Re)$ and of the functions $x_{1c_0}(x_0, \Re)$ that define the sound field in first approximation and the position of the shocks. On the other hand, the relations

/V,22

$$
\left\{
\begin{array}{l}
\left[\tilde{V}_0\left(\tilde{\rho}_0 + \frac{\partial \bar{x}_{1c_0}}{\partial \bar{x}_0}\right)\right] = 0 , \\
\left[\tilde{V}_0\left(\mathbb{D}\, \bar{x}_{1c_0} + \tilde{V}_{\P_0}\right)\right] = 0 , \\
\left[\tilde{V}_{10} \tilde{S}_0\right] = 0
\end{array}
\right. \tag{76}
$$

must ensure, in principle and on the basis of eqs.(60) and (61), a determination of $\tilde{\rho}_0$.

<u>Theorem 2</u>: It is assumed that, in a calm and homogeneous atmosphere, a nonlinear wave train propagates in the vicinity of an acoustic wave obtained by setting $x_1 = 0$ in the representation (16) of theorem 1. Let us assume that the width of the wave train is $O(\ell) \ll O(L)$ where $O(L)$ is the order of magnitude of the distance covered since its generation. By this, we mean that the perturba-

tions are negligible if $x_1 \gg \ell$. Let us select $\epsilon = \frac{\ell}{L}$ as an infinitely small principal term. If the wave train in question admits a limiting structure as $\epsilon \to 0$, this will be obtained as follows: The perturbations $\frac{p - p_0}{\rho_0 c_0^2}$, $\frac{\rho - \rho_0}{\rho_0}$, $\frac{\mathbf{V} \cdot \mathbf{n}}{c_0}$ are of the order ϵ, while the tangential component $\mathbf{V}_T = \mathbf{V} - \mathbf{n}(\mathbf{V} \cdot \mathbf{n})$ of the perturbation velocity is $O(c_0\epsilon^2)$ and the entropy perturbation is $o(c_v\epsilon^3)$. Expressed more accurately, the approximation of the first order for the structure of the wave train is defined by the following formulas:

/V,23

$$\begin{cases} \mathbf{V} = V_1(x_0, x_1; \mathcal{R})\, \mathbf{n}(x_0; \mathcal{R}) , \\ p = p_0 + \rho_0 c_0 V_1 , \\ \rho = \rho_0 + \rho_0 \dfrac{V_1}{c_0} , \\ S = S_0 , \end{cases} \tag{77}$$

where V_1 proves the equation

$$\Gamma_0 \; V_1 \frac{\partial V_1}{\partial x_1} + \frac{\partial V_1}{\partial x_0} + \frac{1}{2} V_1 K = 0 , \tag{78}$$

with $\frac{1}{2}K$ denoting the mean curvature of the wave surface $x_1 = 0$, $x_0 = \text{const}$, counted clockwise if the convexity is in the direction of propagation. The function V_1 is discontinuous on the shock surfaces, if one of these is defined by

$$x_1 = X_{1c}(x_0; \mathcal{R}) , \tag{79}$$

then we have the following condition of shock:

$$\left[V_1 \left(\Gamma_0 V_1 - 2 \frac{\partial X_{1c}}{\partial x_0} \right) \right] = 0 . \tag{80}$$

Using the resultant first approximation, we can improve the description by formulas of the type of

$$\begin{cases} \mathbf{V} = V_1 \mathbf{n} + V_1 \tilde{\mathbf{V}}_T \\ p = p_0 + \rho_0 c_0^2 V_1 (1 + \tilde{p}) \\ \rho = \rho_0 + \rho_0 \dfrac{V_1}{c_0} (1 + \tilde{\rho}) \end{cases} \tag{81}$$

$$\Big\{ \quad S = S_0 + c_v \frac{V_1}{c_0} \tilde{S}$$

by agreeing to denote the true velocity components by V_1 and $V_1 \tilde{V}_T$. The /V,24
magnitude of the limit reached by $\tilde{p}/\epsilon$ and $\tilde{\rho}/\epsilon$ and also by $\tilde{S}/\epsilon^2$ can be calculated uniquely by means of the first approximation obtained for V_1 by eqs.(78) to (80). For this, the following differential system must be used:

$$\left\{ \begin{array}{l} \dfrac{\partial V_1 \tilde{p}}{\partial x_1} + \dfrac{\partial V_1}{\partial x_0} = 0, \\[2ex] \dfrac{\partial V_1 \tilde{V}_T}{\partial x_1} - \mathbb{D} V_1 = 0, \\[2ex] \dfrac{\partial \tilde{S}_0 V_1}{\partial x_1} = 0, \qquad \qquad \tilde{p} = \tilde{\rho} + (\Gamma_0 - 1) \dfrac{V_1}{c_0} \end{array} \right. \tag{82}$$

and the following conditions of shock:

$$\left\{ \begin{array}{ll} \left[V_1 \left(\tilde{\rho} + \dfrac{\partial x_{1c}}{c_0 \partial x_0} \right) \right] = 0, & a) \\[2ex] \left[V_1 \left(c_0 \mathbb{D} x_{1c} + \tilde{V}_T \right) \right] = 0, & b) \\[2ex] \left[c_v T_0 V_1 \tilde{S} - \dfrac{\Gamma}{6} V_1^3 \right] = 0 . & c) \end{array} \right. \tag{83}$$

The calculation that yields the last relation is entirely classical: If the entropy jump $\left[c_v \dfrac{V_1}{c_0} \tilde{S} \right]$ across the shock is to be calculated, one can consider this shock, in first approximation for the calculation of the jump in question, as a normal shock and use, for the intensity of this latter /V,25
shock, the pressure surge $[\rho_0 \tau_0 V_1]$. Denoting by $\overline{f} = f_1 + f_2$ the sum of the values of f upstream and downstream, we will have

$$2T[S] = \overline{\tau} [p] , \tag{84}$$

because of the Hugoniot relation; however, the equation of state of the gas

yields

$$2[h] = \left(2\tau_1 + (\tau_p)_1 [p] + \frac{1}{3}(\tau_{pp})_1 [p]^2\right)[p] + 2T_1 [S] , \tag{85}$$

such that we obtain

$$\hat{T}[S] = \frac{1}{12}\, \overline{\tau_{pp}}\, [p]^3 \left\{1 + O\left(\left[\frac{p}{\rho_0 c_0^2}\right]\right)\right\}, \tag{86}$$

from which eq.(83c) is derived.

5.2 Dispersion, Attenuation, and Convection of Sound in Plane Waves of Finite Amplitude

In this Section, we will discuss the nondimensional and nonstationary flow of a viscous and heat-conducting gas, using the hypothesis in which the Reynolds number, characterizing the relative significance of the effects of an ideal fluid and a dissipative fluid, is very large. We are using the equation of state in the form of $p = g(\tau, s)$ where $\tau = \rho^{-1}$ and denote by μ, μ_v, k the two viscosity coefficients and the conductivity coefficient (see Section 1.1.1).

5.2.1 Equations of Small Perturbations

We start from the equations of motion which are all supposed to be constant /V,26

$$\begin{cases} \dfrac{\partial \tau}{\partial t} + u\dfrac{\partial \tau}{\partial x} - \tau\dfrac{\partial u}{\partial x} = 0 , \\ \dfrac{\partial u}{\partial t} + u\dfrac{\partial u}{\partial x} + \tau g_\tau \dfrac{\partial \tau}{\partial x} + \tau g_s \dfrac{\partial s}{\partial x} = \left(\dfrac{4}{3}\mu + \mu_v\right)\tau\dfrac{\partial^2 u}{\partial x^2} , \\ \dfrac{\partial s}{\partial t} + u\dfrac{\partial s}{\partial x} = \left(\dfrac{4}{3}\mu + \mu_v\right)\tau T^{-1}\left(\dfrac{\partial u}{\partial x}\right)^2 + \dfrac{\tau}{T}\dfrac{\partial}{\partial x}\left(k\dfrac{\partial T}{\partial x}\right) . \end{cases} \tag{1}$$

Let us then introduce

$$P^{(+,-)}(\tau, u, s) = \pm u - \int_{\tau_0}^{\tau} \frac{c(\tau', s)}{\tau'}\, d\tau' , \tag{2}$$

agreeing to fix, once and for all, the origin τ_0. The new dependent variables are P^+, P^-, S, and all quantities must be expressed in terms of these. Then,

eqs.(1) take the following form:

$$
\left\{
\begin{aligned}
&\frac{\partial P^+}{\partial t} + (u+c)\frac{\partial P^+}{\partial x} + \left(\tau g_S - c\frac{\partial P^+}{\partial S}\right)\frac{\partial S}{\partial x} - \frac{\partial P^+}{\partial S}\left(\frac{\partial S}{\partial t} + u\frac{\partial S}{\partial x}\right) = \\
&\qquad\qquad = \frac{1}{2}\left(\frac{4}{3}\mu + \mu_v\right)\tau\,\frac{\partial^2 P^+ - P^-}{\partial x^2}, \\
&\frac{\partial P^-}{\partial t} + (u-c)\frac{\partial P^-}{\partial x} - \left(\tau g_S - c\frac{\partial P^-}{\partial S}\right)\frac{\partial S}{\partial x} - \frac{\partial P^-}{\partial S}\left(\frac{\partial S}{\partial t} + u\frac{\partial S}{\partial x}\right) = \\
&\qquad\qquad = \frac{1}{2}\left(\frac{4}{3}\mu + \mu_v\right)\tau\,\frac{\partial^2 P^- - P^+}{\partial x^2}, \\
&\frac{\partial S}{\partial t} + u\frac{\partial S}{\partial x} = \frac{1}{4}\left(\frac{4}{3}\mu + \mu_v\right)\tau\,T^{-1}\left(\frac{\partial\, P^+ - P^-}{\partial x}\right)^2 + \tau T^{-1}\frac{\partial}{\partial x}\left(k\frac{\partial T}{\partial x}\right).
\end{aligned}
\right.
\tag{3}
$$

Let us assume that the gas is very little perturbed with respect to a reference state of rest which latter we characterize by the subscript o ($\tau = \tau_0$, $S = S_0$, $u = 0$, $P_0^+ = P_0^- = 0$). In the perturbed state, we pose /V,27

$$
P^+ = c_0 U \qquad P^- = c_0 V \qquad S = S_0 + c_v \Sigma \tag{4}
$$

from which we derive

$$
\left\{
\begin{aligned}
u &= \frac{c_0}{2}(U - V) \\
\tau &= \tau_0 - \frac{\tau_0 c_0}{2c_0}(U + V) \\
T &= T_0 + \frac{c_0^2}{2\chi_0 c_{v_0}}(U + V) + \frac{c_0^2}{\chi_0(\lambda_0 - 1)c_{v_0}}\Sigma
\end{aligned}
\right.
\tag{5}
$$

under the condition that $de = -p\,d\tau + TdS$ entrains $T_\tau = -g_S$; we then pose

$$
\tau g_S = +\frac{c^2}{\chi c_v} \qquad\qquad c_v T_S = +\frac{\tau g_S}{\lambda - 1} \tag{6}
$$

in such a manner that $\chi = \lambda = \gamma$ for an ideal gas. After substitution in eqs.(3) and retaining only terms linear in U, V, Σ, we will obtain (μ, ... const)

$$\left\{\begin{aligned} &\frac{\partial U}{\partial t} + c_0 \frac{\partial U}{\partial x} + \frac{c_0}{\chi} \frac{\partial \Sigma}{\partial x} = \frac{1}{2}\left(\frac{4}{3}\mu_0 + \mu_{v_0}\right)\tau_0 \frac{\partial^2 (U-V)}{\partial x^2}, \\ &\frac{\partial V}{\partial t} - c_0 \frac{\partial V}{\partial x} - \frac{c_0}{\chi} \frac{\partial \Sigma}{\partial x} = \frac{1}{2}\left(\frac{4}{3}\mu_0 + \mu_{v_0}\right)\tau_0 \frac{\partial^2 (V-U)}{\partial x^2}, \\ &\frac{\partial \Sigma}{\partial t} = \frac{c_0^2}{c_{v_0} T_0} \frac{k_0 \tau_0}{\chi_0 c_{v_0}} \left\{ \frac{1}{2} \frac{\partial^2 (U+V)}{\partial x^2} + \frac{1}{\lambda_0 - 1} \frac{\partial^2 \Sigma}{\partial x^2} \right\}. \end{aligned}\right. \tag{7}$$

Let us introduce a length scale and a time scale by posing

$$t = \frac{L}{c_0} x_0, \qquad x = L x_1, \tag{8}$$

and let us assume that we wish to return to the linearized form of eqs.(1) with sources of mass, momentum, and energy; this will yield

/V,28

$$\left\{\begin{aligned} &\frac{\partial T}{\partial t} - \tau_0 \frac{\partial u}{\partial x} = -\frac{c_0 \tau_0}{L} Q, \\ &\frac{\partial u}{\partial t} + \tau_0 g_{T_0} \frac{\partial T}{\partial x} + \tau_0 g_{S_0} \frac{\partial S}{\partial x} - \left(\frac{4}{3}\mu_0 + \mu_{v_0}\right)\tau_0 \frac{\partial^2 u}{\partial x^2} = \frac{c_0^2}{L} M, \\ &\frac{\partial S}{\partial t} - \frac{\tau_0 k_0}{T_0} \frac{\partial^2 T}{\partial x^2} = \frac{c_0}{L} \frac{E}{c_v}, \end{aligned}\right. \tag{9}$$

i.e., on returning to eq.(7)

$$\left\{\begin{aligned} &\frac{\partial U}{\partial x_0} + \frac{\partial U}{\partial x_1} + \frac{1}{\chi} \frac{\partial \Sigma}{\partial x_1} - \frac{1}{2}\left(\frac{4}{3}\mu_0 + \mu_{v_0}\right)\tau_0 L_0^{-1} c_0^{-1} \frac{\partial^2 (U-V)}{\partial x_1^2} = M + Q, \\ &\frac{\partial V}{\partial x_0} - \frac{\partial V}{\partial x_1} - \frac{1}{\chi} \frac{\partial \Sigma}{\partial x_1} - \frac{1}{2}\left(\frac{4}{3}\mu_0 + \mu_{v_0}\right)\tau_0 L_0^{-1} c_0^{-1} \frac{\partial^2 (V-U)}{\partial x_1^2} = M - Q, \\ &\frac{\partial \Sigma}{\partial x_0} - \frac{c_0^2}{c_v T_0} \frac{k_0 \tau_0}{\chi_0 c_{v_0} L c_0} \left\{ \frac{1}{2} \frac{\partial^2 (U+V)}{\partial x_1^2} + \frac{1}{\lambda - 1} \frac{\partial^2 \Sigma}{\partial x_1^2} \right\} = E, \end{aligned}\right. \tag{10}$$

where $\mathfrak{M}$, Q, E are, in dimensionless form, the densities of the mass sources, of the momentum, and of the heat. We will assume that these sources vanish outside

of a domain bounded by the plane x_0, x_1.

Let us use the Laplace transformation

$$\hat{f}(\zeta_0, \zeta_1) = \iint_{-\infty}^{\infty} e^{-(\zeta_0 x_0 + \zeta_1 x_1)} f(x_0, x_1)\, dx_0\, dx_1 \tag{11}$$

with

$$\zeta_0 = \xi_0 + i\eta_0 \qquad \zeta_1 = \xi_1 + i\eta_1 \tag{12}$$

and let us pose

$$\begin{cases} \left(\frac{4}{3}\mu_0 + \mu_{V_0}\right)\tau_0 L^{-1} c_0^{-1} = \varepsilon, \\ \dfrac{c_0^2}{C_V T_0}\,\dfrac{k_0 \tau_0}{\chi c_{V_0} L c_0} = \kappa\varepsilon \end{cases} \tag{13}$$

by noting that ϵ is the inverse of a Reynolds number and that, consequently, /V,29

$$\varepsilon \ll 1 \tag{14}$$

occurs in the contemplated applications, while K is correlated to the Prandtl number and remains of the order of unity.

After transformation, the system (10) becomes

$$\begin{cases} \left(\zeta_0 + \zeta_1 - \frac{1}{2}\varepsilon\zeta_1^2\right)\hat{U} + \frac{1}{2}\varepsilon\zeta_1^2\hat{V} + \dfrac{\zeta_1}{\chi}\hat{\Sigma} = \hat{M} + \hat{\Phi}, \\ \left(\zeta_0 - \zeta_1 - \frac{1}{2}\varepsilon\zeta_1^2\right)\hat{U} + \frac{1}{2}\varepsilon\zeta_1^2\hat{V} - \dfrac{\zeta_1}{\chi}\hat{\Sigma} = \hat{M} - \hat{\Phi}, \\ \left(\zeta_0 - \dfrac{\varepsilon\kappa}{\lambda-1}\zeta_1^2\right)\hat{\Sigma} - \frac{1}{2}\varepsilon\kappa\zeta_1^2\left(\hat{U} + \hat{V}\right) = \hat{E}, \end{cases} \tag{15}$$

which is a system linear in $\hat{U}$, $\hat{V}$, $\hat{\Sigma}$ having the following determinant

$$\hat{\Delta}(\zeta_0, \zeta_1; \varepsilon) \equiv \left(\zeta_0^2 - \zeta_1^2 - \varepsilon\zeta_0\zeta_1^2\right)\left(\zeta_0 - \frac{\varepsilon\kappa}{\lambda-1}\zeta_1^2\right) - \frac{\varepsilon\kappa}{\chi}\zeta_1^4 \tag{16}$$

whose solution is expressed in the form of

$$\left\{\begin{aligned}
\Delta\hat{U} &= \hat{\mathfrak{M}}\,\hat{P}_{\hat{\mathfrak{M}},\hat{U}}(\zeta_0,\zeta_1;\varepsilon) + \hat{Q}\,\hat{P}_{\hat{Q},\hat{U}}(\zeta_0,\zeta_1;\varepsilon) + \hat{E}\,\hat{P}_{\hat{E},\hat{U}}(\zeta_0,\zeta_1;\varepsilon)\\
\Delta\hat{V} &= \hat{\mathfrak{M}}\,\hat{P}_{\hat{\mathfrak{M}},\hat{V}}(\zeta_0,\zeta_1;\varepsilon) + \hat{Q}\,\hat{P}_{\hat{Q},\hat{V}}(\zeta_0,\zeta_1;\varepsilon) + \hat{E}\,\hat{P}_{\hat{E},\hat{V}}(\zeta_0,\zeta_1;\varepsilon)\\
\Delta\hat{\Sigma} &= \hat{\mathfrak{M}}\,\hat{P}_{\hat{\mathfrak{M}},\hat{\Sigma}}(\zeta_0,\zeta_1;\varepsilon) + \hat{Q}\,\hat{P}_{\hat{Q},\hat{\Sigma}}(\zeta_0,\zeta_1;\varepsilon) + \hat{E}\,\hat{P}_{\hat{E},\hat{\Sigma}}(\zeta_0,\zeta_1;\varepsilon)
\end{aligned}\right. \qquad (17)$$

where all P are polynomials in ζ_0 and ζ_1 dependent on ϵ but independent of $\hat{\mathfrak{M}}$, $\hat{Q}$, $\hat{E}$, i.e.,

/V,30

$$\left\{\begin{aligned}
&\hat{P}_{\hat{\mathfrak{M}},\hat{U}} \equiv (\zeta_0-\zeta_1)\left(\zeta_0-\frac{\varepsilon\kappa}{\lambda-1}\zeta_1^2\right)-\varepsilon\zeta_1^2\left(\zeta_0+\frac{\kappa}{\chi}\zeta_1-\frac{\varepsilon\kappa}{\lambda-1}\zeta_1^2\right), && a)\\
&\hat{P}_{\hat{Q},\hat{U}} \equiv (\zeta_0-\zeta_1)\left(\zeta_0-\frac{\varepsilon\kappa}{\lambda-1}\zeta_1^2\right), && b)\\
&\hat{P}_{\hat{E},\hat{U}} \equiv -\frac{\zeta_1}{\chi}(\zeta_0-\zeta_1), && c)
\end{aligned}\right. \qquad (18)$$

$$\left\{\begin{aligned}
&\hat{P}_{\hat{\mathfrak{M}},\hat{V}} \equiv (\zeta_0+\zeta_1)\left(\zeta_0-\frac{\varepsilon\kappa}{\lambda-1}\zeta_1^2\right)-\varepsilon\zeta_1^2\left(\zeta_0-\frac{\kappa}{\chi}\zeta_1-\frac{\varepsilon\kappa}{\lambda-1}\zeta_1^2\right), && a)\\
&\hat{P}_{\hat{Q},\hat{V}} \equiv -(\zeta_0+\zeta_1)\left(\zeta_0-\frac{\varepsilon\kappa}{\lambda-1}\zeta_1^2\right), && b)\\
&\hat{P}_{\hat{E},\hat{V}} \equiv \frac{\zeta_1}{\chi}(\zeta_0+\zeta_1), && c)
\end{aligned}\right. \qquad (19)$$

$$\left\{\begin{aligned}
&\hat{P}_{\hat{\mathfrak{M}},\hat{\Sigma}} \equiv \varepsilon\kappa\zeta_1^2(\zeta_0-\varepsilon\zeta_1^2), && a)\\
&\hat{P}_{\hat{Q},\hat{\Sigma}} \equiv -\varepsilon\kappa\zeta_1^3, && b)\\
&\hat{P}_{\hat{E},\hat{\Sigma}} = \zeta_0^2-\zeta_1^2-\varepsilon\zeta_0\zeta_1^2. && c)
\end{aligned}\right. \qquad (20)$$

Returning to the originals, expressions of the following type must be treated:

$$I = \frac{1}{(2\pi)^2} \int_{\xi_0 - i\infty}^{\xi_0 + i\infty} d\eta_0 \int_{\xi_1 - i\infty}^{\xi_1 + i\infty} d\eta_1 \; \frac{\hat{F}(\zeta_0, \zeta_1) \; \hat{\underset{\sim}{P}}(\zeta_0, \zeta_1)}{\hat{\Delta}(\zeta_0, \zeta_1 ; \varepsilon)} \, e^{(x_0\zeta_0 + x_1\zeta_1)} \tag{21}$$

where F is an integral function, Δ is the function (16), and $\underset{\sim}{P}$ is one of the polynomials (18) - (20). Assuming that Q, $\mathfrak{M}$, E are sufficiently regular, we obtain an augmentation of the following type:

$$|\hat{F}| < C\left(1 + |\zeta_1|^2 + |\zeta_0|^2\right)^{-N/2} \exp\left\{R\left(\xi_0^2 + \xi_1^2\right)^{\frac{1}{2}}\right\}, \tag{22}$$

so that there will be no difficulty of convergence and that the main problem will concern the zeros of Δ. If $\xi_1 = 0$, these are the three roots of the /V,31
equation of the third degree in ζ_0

$$\left[\zeta_0^2 + \eta_1^2(1 + \varepsilon\zeta_0)\right]\left(\zeta_0 + \frac{\varepsilon K}{\lambda - 1}\eta_1^2\right) - \frac{\varepsilon K}{\chi}\eta_1^4 = 0 \; . \tag{23}$$

If ϵ is very small, these roots will be approximately given by

$$\left\{ \begin{aligned} \zeta_0 &= \varepsilon K \eta_1^2 \, \frac{\lambda - \chi - 1}{\chi(\lambda - 1)} \left\{1 - \frac{\varepsilon^2 K}{\chi}\eta_1^2 + \cdots\right\} \\ \zeta_0 &= \pm i\eta_1 - \frac{\varepsilon}{2}\left(\frac{K}{\chi} + 1\right)\eta_1^2 + \cdots . \end{aligned} \right. \tag{24}$$

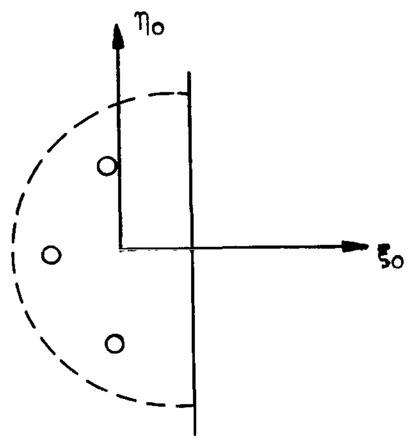
However, this approximation no longer holds for large values of η_1. Without approximation, the real roots of eq.(23) will verify

$$\underset{\sim}{P}(\xi_0) \equiv \xi_0^3 + \frac{\varepsilon(K + \lambda - 1)}{\lambda - 1}\eta_1^2 \xi_0^2 + \eta_1^2\left(1 + \frac{\varepsilon^2 K}{\lambda - 1}\eta_1^2\right)\xi_0 = -\frac{\varepsilon K}{\lambda - 1}\eta_1^4 + \frac{\varepsilon K}{\chi}\eta_1^4 \; . \tag{25}$$

For an ideal gas, $\chi = \lambda = \gamma$ and the right-hand side of the equation will become $-\frac{\epsilon K}{\gamma(\gamma - 1)}\eta_1^4 = -\frac{\epsilon}{4/3 \; Pr + Pr_v}\eta_1^4$ with $Pr = \frac{\mu c_p}{K}$, $Pr_v = \frac{\mu_v c_p}{K}$; we stipulate here that $\lambda - 1 - \chi$ is generally negative. If this is the case, a real root of eq.(23) must render the polynomial $\underset{\sim}{P}(\xi_0)$ also negative. However, this latter has no positive root and it even can be stated that it only has a single real root if the following condition is satisfied:

$$\varepsilon^2 \eta_1^2 \frac{K+\lambda-1}{\lambda-1} \frac{K-3(\lambda-1)}{\lambda-1} < 4 , \tag{26}$$

which certainly takes place for very small ϵ if ζ_1 is not very large. In /V,32 the case of an ideal monatomic gas, we have $\frac{K}{\lambda - 1} - 3 = \frac{3\gamma}{4Pr} - 3 < 0$ and eq.(26) takes place for all η_1. Thus, under sufficiently extensive hypotheses as to the



values of K, λ, χ, eq.(23) admits one negative real root and two imaginary roots conjugate to the negative real part. If ϵ is small, an approximate representation of these roots is given by eq.(24), except for large values of η_1.

Let us return to eq.(21) and let us first explicitly write the integration with respect to η_0 which will be calculated by means of the method of residues, thus yielding

$$I = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\eta_1 \sum{}^{*} \frac{e^{x_0 \zeta_0^* + x_1 \zeta_1} \hat{F}(\zeta_0^*, i\eta_1)\, \hat{P}(\zeta_0^*, i\eta_1)}{\frac{\partial \hat{\Delta}(\zeta_0^*, i\eta_1)}{\partial \zeta_0}} , \tag{27}$$

where the sum Σ^* is extended to three roots of eq.(23) which, in agreement with the preceding result, can be denoted by $\zeta_0^{(\epsilon)}$, $\zeta_0^{(+)}$, and $\zeta_0^{(-)}$ for designating, respectively, the real root and for differentiating, among the two conjugate imaginary roots, the root which has a positive imaginary part from the root which has a negative imaginary part. To each of the roots $\zeta_0^{(\epsilon)}$, $\zeta_0^{(+,-)}$, an integral $I^{(\epsilon)}$, $I^{(+,-)}$ can be added so that we obtain

$$I = I^{(+)} + I^{(-)} + I^{(\epsilon)} . \tag{28}$$

We will then attempt to interpret $\hat{I}^{(+)}$, $\hat{I}^{(-)}$, and $\hat{I}^{(\varepsilon)}$. For this, let us recall that $\hat{F}$ denotes one of the functions $\mathfrak{M}$, $\mathfrak{Q}$, $\hat{E}$ while $\hat{P}$ denotes one of the polynomials (18), (19), (20). According to eq.(22), we have /V,33

$$|\hat{F}| < \frac{C \exp(R|\xi_0|)}{(1+|\eta_1|^2+|\zeta_0^*|^2)^{N/2}}, \tag{29}$$

so that, if N is sufficiently large and if $x_0 > R$, then the quantity under the integral sign in eq.(27) can be augmented by an expression of the type C_1 $(1 + |\eta_1|^2)^{M/2}$ where C_1 is independent of ϵ, from which we can conclude that it is legitimate to determine the limiting behavior of I as $\epsilon \to 0$, by passing to the limit under the integral sign. In addition, if ϵ is very small for a large value of η_1, the real parts of the roots $\zeta_0^{(\varepsilon+,-)}$ are large in absolute value and are negative, such that, for $x_0 > R$, the exponential sharply attenuates the large values of η_1; thus, we cannot only pass to the limit under the integral sign but even can perform an asymptotic expansion in powers of ϵ. Taking

$\frac{\partial\Delta}{\partial\zeta_0} = 3\zeta_0^2 + 2\frac{\epsilon(K+\lambda-1)}{\lambda-1}\eta_1^2\zeta_0 + \eta_1^2(1+\frac{\epsilon K}{\lambda-1}\eta_1^2)$ into consideration, we have

$$I^{(+,-)} \cong -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(x_1\eta_1 \pm x_0\eta_1) - \frac{\epsilon}{2}(\frac{K}{\chi}+1)\eta_1^2 x_0} \frac{\hat{F}(\pm i\eta_1,\eta_1)\,\hat{P}(\pm i\eta_1,\eta_1)}{2\eta_1^2}\,d\eta_1$$

$$I^{(\epsilon)} \cong \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ix_1\eta_1 - \frac{\epsilon K(\chi-\lambda+1)}{\chi(\lambda-1)}\eta_1^2 x_0}\,\frac{\hat{F}(0,\eta_1)\,\hat{P}(0,\eta_1)}{\eta_1^2}\,d\eta_1 \tag{29}$$

Writing $\hat{P}$ and $\hat{F}$ in explicit form, we can consider the expressions $I_{\mathfrak{M},U}^{(+,-,e)}$, ...; we can then state that /V,34

$$I_U^{(+)} = I_{M,U}^{(+)} + I_{Q,U}^{(+)} + I_{E,U}^{(+)}, \tag{30}$$

characterizes the first mode of acoustic propagation of U, whereas

$$I_U^{(-)} = I_{M,U}^{(-)} + I_{Q,U}^{(-)} + I_{E,U}^{(-)}, \tag{31}$$

characterizes the second acoustic mode, while

$$I_U^{(e)} = I_{M,U}^{(e)} + I_{Q,U}^{(e)} + I_{E,U}^{(e)}, \tag{32}$$

characterizes the entropy mode. The same classification goes for V and E instead of U. In particular, we have the following:

Acoustic modes:

$$
\left\{
\begin{aligned}
I^{(+)}_{\mathfrak{M},U} &\cong -\frac{\varepsilon}{2\pi}\,\frac{K+\chi}{4\chi}\int_{-\infty}^{\infty} e^{i(x_1+x_0)\eta_1-\frac{\varepsilon}{2}\left(\frac{K}{\chi}+1\right)\eta_1^2 x_0}\,\hat{\mathfrak{M}}(i\eta_1, i\eta_1)\, i\eta_1\, d\eta_1 \\
I^{(+)}_{\mathcal{Q},U} &\cong \frac{\varepsilon}{2\pi}\,\frac{K+\chi}{4\chi}\int_{-\infty}^{\infty} e^{i(x_1+x_0)\eta_1-\frac{\varepsilon}{2}\left(\frac{K}{\chi}+1\right)\eta_1^2 x_0}\,\hat{\mathcal{Q}}(i\eta_1, i\eta_1)\, i\eta_1\, d\eta_1 \\
I^{(+)}_{E,U} &\cong -\frac{\varepsilon}{2\pi}\,\frac{K+\chi}{4\chi^2}\int_{-\infty}^{\infty} e^{i(x_1+x_0)\eta_1-\frac{\varepsilon}{2}\left(\frac{K}{\chi}+1\right)\eta_1^2 x_0}\,\hat{E}(i\eta_1, i\eta_1)\, i\eta_1\, d\eta_1 \\
I^{(+)}_{\mathfrak{M},V} &\cong \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(x_1+x_0)\eta_1-\frac{\varepsilon}{2}\left(\frac{K}{\chi}+1\right)\eta_1^2 x_0}\,\hat{\mathfrak{M}}(+i\eta_1, i\eta_1)\, d\eta_1 \\
I^{(+)}_{\mathcal{Q},V} &\cong -\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(x_1+x_0)\eta_1-\frac{\varepsilon}{2}\left(\frac{K}{\chi}+1\right)\eta_1^2 x_0}\,\hat{\mathcal{Q}}(+i\eta_1, i\eta_1)\, d\eta_1 \\
I^{(+)}_{E,V} &\cong \frac{1}{2\pi\chi}\int_{-\infty}^{\infty} e^{i(x_1+x_0)\eta_1-\frac{\varepsilon}{2}\left(\frac{K}{\chi}+1\right)\eta_1^2 x_0}\,\hat{E}(+i\eta_1, i\eta_1)\, d\eta_1 \\
I^{(+)}_{\mathfrak{M},\Sigma} &\cong +\frac{\varepsilon K}{2}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(x_1+x_0)\eta_1-\frac{\varepsilon(K+\chi)}{2\chi}\eta_1^2 x_0}\,\hat{\mathfrak{M}}(i\eta_1, i\eta_1)\, i\eta_1\, d\eta_1 \\
I^{+}_{\mathcal{Q},\Sigma} &\cong -\frac{\varepsilon K}{2}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(x_1+x_0)\eta_1-\frac{\varepsilon}{2}\left(\frac{K}{\chi}+1\right)\eta_1^2 x_0}\,\hat{\mathcal{Q}}(i\eta_1, i\eta_1)\, i\eta_1\, d\eta_1 \\
I^{+}_{E,\Sigma} &\cong +\frac{\varepsilon K}{2\chi}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i(x_1+x_0)\eta_1-\frac{\varepsilon}{2}\left(\frac{K}{\chi}+1\right)\eta_1^2 x_0}\,\hat{E}(i\eta_1, i\eta_1)\, i\eta_1\, d\eta_1
\end{aligned}
\right.
\tag{33}
$$

/V,35

The formulas relative to $I^{(-)}$ are obtained on substituting i by -i and x_1 by $-x_1$ (Note: $\hat{\mathfrak{M}}(i\eta_1, in_1) \to \hat{\mathfrak{M}}(-i\eta_1, in_1)$, etc.).

Entropy mode:

$$
\left\{
\begin{aligned}
I^{(e)}_{\mathfrak{M},U} &\cong -\frac{\varepsilon K}{\lambda-1}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i x_1 \eta_1 - \varepsilon K \frac{(\varkappa-\lambda+1)}{\varkappa(\lambda-1)}\eta_1^2 x_0}\,\hat{\mathfrak{M}}(0,i\eta_1)\, i\eta_1\, d\eta_1 \\
I^{(e)}_{Q,U} &\cong \varepsilon K\left(\frac{1}{2\varkappa}-\frac{1}{\lambda-1}\right)\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i x_1 \eta_1 - \frac{\varepsilon K(\varkappa-\lambda+1)}{\varkappa(\lambda-1)}\eta_1^2 x_0}\,\hat{Q}(0,i\eta_1)\, i\eta_1\, d\eta_1 \\
I^{(e)}_{E,U} &\cong \frac{1}{2\varkappa}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i x_1 \eta_1 - \frac{\varepsilon K(\varkappa-\lambda+1)}{\varkappa(\lambda-1)}\eta_1^2 x_0}\,\hat{E}(0,i\eta_1)\, d\eta_1 \\
I^{(e)}_{\mathfrak{M},V} &\cong \frac{\varepsilon K}{\lambda-1}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i x_1 \eta_1 - \frac{\varepsilon K(\varkappa-\lambda+1)}{\varkappa(\lambda-1)}\eta_1^2 x_0}\,\hat{\mathfrak{M}}(0,i\eta_1)\, i\eta_1\, d\eta_1 \\
I^{(e)}_{Q,V} &\cong \varepsilon K\left(\frac{1}{2\varkappa}-\frac{1}{\lambda-1}\right)\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i x_1 \eta_1 - \frac{\varepsilon K(\varkappa-\lambda+1)}{\varkappa(\lambda-1)}\eta_1^2 x_0}\,\hat{Q}(0,i\eta_1)\, i\eta_1\, d\eta_1 \\
I^{(e)}_{E,V} &\cong \frac{1}{2\varkappa}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i x_1 \eta_1 - \frac{\varepsilon K(\varkappa-\lambda+1)}{\varkappa(\lambda-1)}\eta_1^2 x_0}\,\hat{E}(0,i\eta_1)\, d\eta_1 \\
I^{(e)}_{\hat{\mathfrak{M}},\Sigma} &\cong -\frac{\varepsilon^2 K}{2}\left\{\frac{K(\varkappa-\lambda+1)}{\varkappa(\lambda-1)}+1\right\}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i x_1 \eta_1 - \frac{\varepsilon K(\varkappa-\lambda+1)}{\varkappa(\lambda-1)}\eta_1^2 x_0}\,\eta_1^2\,\hat{\mathfrak{M}}(0,i\eta_1)\, d\eta_1 \\
I^{(e)}_{Q,\Sigma} &\cong 2\,\frac{\varepsilon K}{2}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i x_1 \eta_1 - \frac{\varepsilon K(\varkappa-\lambda+1)}{\varkappa(\lambda-1)}\eta_1^2 x_0}\, i\eta_1\,\hat{Q}(0,i\eta_1)\, d\eta_1 \\
I^{(e)}_{E,\Sigma} &\cong 2\,\frac{1}{2}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i x_1 \eta_1 - \varepsilon K\frac{(\varkappa-\lambda+1)}{\varkappa(\lambda-1)}\eta_1^2 x_0}\,\hat{E}(0,i\eta_1)\, d\eta_1 \, .
\end{aligned}
\right. \tag{34}
$$

An investigation of eqs.(33) and (34) demonstrates that we basically have two types of integrals to calculate: /V,36

$$
\left\{
\begin{aligned}
J^{(a)}_{f} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i z \eta_1 - \varepsilon\theta\eta_1^2}\,\hat{f}(\pm i\eta_1, i\eta_1)\, d\eta_1 \, , \\
J^{(e)}_{f} &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i z \eta_1 - \varepsilon\theta\eta_1^2}\,\hat{f}(0, i\eta_1)\, d\eta_1 \, ,
\end{aligned}
\right. \tag{35}
$$

where

$$\hat{f}(\zeta_0, \zeta_1) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{x_0 \zeta_0 + x_1 \zeta_1} f(x_0, x_1)\, dx_0\, dx_1 . \tag{36}$$

For this, let us introduce the notation

$$\begin{cases} f^{(a)}(z) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} e^{iz\eta_1} \hat{f}(\pm i\eta_1, i\eta_1)\, d\eta_1 , \\ f^{(e)}(z) = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} e^{iz\eta_1} \hat{f}(0, i\eta_1)\, d\eta_1 , \end{cases} \tag{37}$$

and note that we have

$$\begin{aligned} \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iz\eta_1 - \varepsilon\theta\eta_1^2} d\eta_1 &= \frac{e^{-z^2/4\varepsilon\theta}}{2\pi} \int_{-\infty}^{\infty} e^{-\varepsilon\theta\left(\eta_1 - \frac{iz}{2\varepsilon\theta}\right)^2} d\eta_1 \\ &= \frac{e^{-z^2/4\varepsilon\theta}}{\sqrt{4\varepsilon\pi\theta}} , \end{aligned} \tag{38}$$

such that $J_f^{(a)}$ and $J_f^{(e)}$ are immediately expressed in the form of

$$J_f^{(a,e)}(z, \theta) = \int_{-\infty}^{\infty} f^{(a)}(u) \frac{e^{-(z-u)^2/4\varepsilon\theta}}{\sqrt{4\pi\varepsilon\theta}}\, du , \tag{39}$$

which solves the problem if we are able to write $f^{(a)}(z)$ and $f^{(e)}(z)$ in explicit form. So far as $f^{(e)}(z)$ is concerned, this is immediate in view of the fact that /V,37

$$\hat{f}(0, i\eta_1) = \int_{-\infty}^{\infty} e^{-i\eta_1 x_1} dx_1 \int_{-\infty}^{\infty} f(x_0, x_1)\, dx_0 , \tag{40}$$

from which, by applying the Fourier reciprocity theorem, we obtain

$$f^{(e)}(z) = \int_{-\infty}^{\infty} f(x_0, z)\, dx_0 \quad . \tag{41}$$

Let us, from this, go to $f^{(a)}(z)$ by defining $f^{(a,+)}$ or $f^{(a,-)}$, yielding

$$f^{(a,\pm)}(z) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{iz\eta_1}\, d\eta_1 \iint_{-\infty}^{\infty} e^{-i\eta_1(x_1 \pm x_0)} f(x_0, x_1)\, dx_0\, dx_1 \,, \tag{42}$$

which leads us to pose

$$\begin{cases} x_1 + x_0 = \lambda^+ \,, \qquad x_1 - x_0 = \lambda^- \,, \\ f(x_0, x_1) = \tilde{f}(\lambda^+, \lambda^-) \,, \end{cases} \tag{43}$$

by means of which the reciprocity theorem yields

$$\begin{cases} f^{(a,\pm)}(z) = \frac{1}{4\pi}\int_{-\infty}^{\infty} e^{iz\eta_1}\, d\eta_1 \int_{-\infty}^{\infty} e^{-i\eta_1 \lambda^{(+,-)}}\, d\lambda^{(+,-)} \int_{-\infty}^{\infty} \tilde{f}(\lambda^+, \lambda^-)\, d\lambda^{(-,+)} \,, \\ f^{(a,+)}(z) = \frac{1}{2}\int_{-\infty}^{\infty} \tilde{f}(z, \lambda^-)\, d\lambda^- \,, \\ f^{(a,-)}(z) = \frac{1}{2}\int_{-\infty}^{\infty} \tilde{f}(\lambda^+, z)\, d\lambda^+ \,. \end{cases} \tag{44}$$

On assembling the results, we obtain

$$U = \int_{-\infty}^{\infty} d\xi \, \frac{\exp\{-(x_1 + x_0 - \xi)^2 / 2\varepsilon(\frac{K}{\chi}+1)x_0\}}{\sqrt{2\pi\varepsilon(K/\chi + 1)x_0}} \times$$

$$\times \left\{ \frac{\varepsilon(K+\chi)}{8\chi} \frac{\partial}{\partial \lambda^+} \int_{-\infty}^{\infty} \left[\tilde{Q}(\lambda^+, \lambda^-) - \tilde{M}(\lambda^+, \lambda^-) - \chi^{-1}\tilde{E}(\lambda^+, \lambda^-) \right] d\lambda^- \right\}_{\lambda^+ = \xi}$$

(continued)

$$+\int_{-\infty}^{\infty} d\xi\, \frac{\exp\{-(x_1-x_0-\xi)^2/2\epsilon(\frac{K}{\chi}+1)x_0\}}{\sqrt{2\pi\epsilon(K/\chi+1)x_0}} \left\{\frac{1}{2}\int_{-\infty}^{\infty}[\tilde{M}(\lambda^+,\lambda^-)+\tilde{Q}(\lambda^+,\lambda^-)+\chi^{-1}\tilde{E}(\lambda^+,\lambda^-)]\,d\lambda^+\right\}_{\lambda^-=\xi}$$

$$+\int_{-\infty}^{\infty} d\xi\, \frac{\exp\{-(x_1-\xi)^2/4\epsilon\frac{K(\chi-\lambda+1)}{\chi(\lambda-1)}x_0\}}{\sqrt{\frac{4\pi\epsilon K(\chi-\lambda+1)}{\chi(\lambda-1)}x_0}} \left\{\int_{-\infty}^{\infty} dx_0'\left[\chi^{-1}E(x_0',\xi)+\epsilon K\left(\frac{1}{\chi}-\frac{2}{\lambda-1}\right)\frac{\partial Q(x_0',\xi)}{\partial x_1}-\frac{2\epsilon K}{\lambda-1}\frac{\partial M(x_0',\xi)}{\partial x_1}\right]\right\}$$

$$V=\int_{-\infty}^{\infty} d\xi\, \frac{\exp\{-(x_1+x_0-\xi)^2/2\epsilon(\frac{K}{\chi}+1)x_0\}}{\sqrt{2\pi\epsilon(K/\chi+1)x_0}} \left\{\frac{1}{2}\int_{-\infty}^{\infty}[\tilde{M}(\lambda^+,\lambda^-)-\tilde{Q}(\lambda^+,\lambda^-)+\chi^{-1}\tilde{E}(\lambda^+,\lambda^-)]\,d\lambda^-\right\}_{\lambda^+=\xi}$$

$$+\int_{-\infty}^{\infty} d\xi\, \frac{\exp\{-(x_1-x_0-\xi)^2/2\epsilon(\frac{K}{\chi}+1)x_0\}}{\sqrt{2\pi\epsilon(K/\chi+1)x_0}} \left\{\frac{\epsilon(K+\chi)}{8\chi}\frac{\partial}{\partial\lambda^-}\int_{-\infty}^{\infty}[\tilde{M}(\lambda^+,\lambda^-)+\tilde{Q}(\lambda^+,\lambda^-)+\chi^{-1}\tilde{E}(\lambda^+,\lambda^-)]\,d\lambda^+\right\}_{\lambda^-=\xi} \qquad (45)$$

$$+\int_{-\infty}^{\infty} d\xi\, \frac{\exp\{-(x_1-\xi)^2/4\epsilon\frac{K(\chi-\lambda+1)}{\chi(\lambda-1)}x_0\}}{\sqrt{\frac{4\pi\epsilon K(\chi-\lambda+1)}{\chi(\lambda-1)}x_0}} \left\{\int_{-\infty}^{\infty} dx_0'\left[\chi^{-1}E(x_0',\xi)+\epsilon K\left(\frac{1}{\chi}-\frac{2}{\lambda-1}\right)\frac{\partial Q(x_0',\xi)}{\partial x_1}+\frac{2\epsilon K}{\lambda-1}\frac{\partial M(x_0',\xi)}{\partial x_1}\right]\right\}$$

$$\Sigma=\int_{-\infty}^{\infty} d\xi\, \frac{\exp\{-(x_1+x_0-\xi)^2/2\epsilon(\frac{K}{\chi}+1)x_0\}}{\sqrt{2\pi\epsilon(K/\chi+1)x_0}} \left\{\frac{\epsilon K}{4}\frac{\partial}{\partial\lambda^+}\int_{-\infty}^{\infty}[\tilde{M}(\lambda^+,\lambda^-)-\tilde{Q}(\lambda^+,\lambda^-)+\chi^{-1}\tilde{E}(\lambda^+,\lambda^-)]\,d\lambda^-\right\}_{\lambda^+=\xi}$$

$$+\int_{-\infty}^{\infty} d\xi\, \frac{\exp\{-(x_1-x_0-\xi)^2/2\epsilon(K/\chi+1)x_0\}}{\sqrt{2\pi\epsilon(K/\chi+1)x_0}} \left\{-\frac{\epsilon K}{4}\frac{\partial}{\partial\lambda^-}\int_{-\infty}^{\infty}[\tilde{M}(\lambda^+,\lambda^-)+\tilde{Q}(\lambda^+,\lambda^-)+\chi^{-1}\tilde{E}(\lambda^+,\lambda^-)]\,d\lambda^+\right\}_{\lambda^-=\xi}$$

$$+\int_{-\infty}^{\infty} d\xi\, \frac{\exp\{-(x_1-\xi)^2/4\epsilon\frac{K(\chi-\lambda+1)}{\chi(\lambda-1)}\}}{\sqrt{4\pi\epsilon\frac{K(\chi-\lambda+1)}{\chi(\lambda-1)}x_0}} \left\{\int_{-\infty}^{\infty} dx_0'\left[E(x_0',\xi)+\epsilon K\frac{\partial}{\partial x_1}Q(x_0',\xi)+\epsilon^2K\frac{\partial^2}{\partial x_1^2}M(x_0',\xi)\right]\right\}$$

If, in the preceding formulas, we retain only the predominant terms, we will obtain /V,39

$$\begin{cases} U = \displaystyle\int_{-\infty}^{\infty} d\xi \, \frac{\exp\{-(x_1+x_0-\xi)^2/2\varepsilon(K/\chi+1)x_0\}}{\sqrt{2\pi\varepsilon(K/\chi+1)x_0}} \left\{\frac{1}{2}\int_{-\infty}^{\infty}[\tilde{\mathfrak{M}}(\lambda^+,\xi)+\tilde{\mathfrak{Q}}(\lambda^+,\xi)+\chi^{-1}\tilde{E}(\lambda^+,\xi)]d\lambda^+\right\} \\ \quad + \displaystyle\int_{-\infty}^{\infty} d\xi \, \frac{\exp\left\{-(x_1-\xi)^2/4\varepsilon\frac{K(\chi-\lambda+1)}{\chi(\lambda-1)}x_0\right\}}{\sqrt{4\pi\varepsilon\frac{K(\chi-\lambda+1)}{\chi(\lambda-1)}x_0}} \int_{-\infty}^{\infty}\chi^{-1}E(x_0',\xi)\,dx_0' \\ V = \displaystyle\int_{-\infty}^{\infty} d\xi \, \frac{\exp\{-(x_1+x_0-\xi)^2/2\varepsilon(K/\chi+1)x_0\}}{\sqrt{2\pi\varepsilon(K/\chi+1)x_0}} \left\{\frac{1}{2}\int_{-\infty}^{\infty}[\tilde{\mathfrak{M}}(\xi,\lambda^-)-\tilde{\mathfrak{Q}}(\xi,\lambda^-)+\chi^{-1}\tilde{E}(\xi,\lambda^-)]d\lambda^-\right\} \\ \quad + \displaystyle\int_{-\infty}^{\infty} d\xi \, \frac{\exp\left\{-(x_1-\xi)^2/4\varepsilon\frac{K(\chi-\lambda+1)}{\chi(\lambda-1)}x_0\right\}}{\sqrt{4\pi\varepsilon\frac{K(\chi-\lambda+1)}{\chi(\lambda-1)}}} \int_{-\infty}^{\infty}\chi^{-1}E(x_0',\xi)\,dx_0' \\ \Sigma = \displaystyle\int_{-\infty}^{\infty} d\xi \, \frac{\exp\left\{-(x_1-\xi)^2/4\varepsilon\frac{K(\chi-\lambda+1)}{\chi(\lambda-1)}x_0\right\}}{\sqrt{4\pi\varepsilon\frac{K(\chi-\lambda+1)}{\chi(\lambda-1)}x_0}} \int_{-\infty}^{\infty}E(x_0',\xi)\,dx_0'. \end{cases} \tag{46}$$

Let us note the following values for an ideal gas:

$$\begin{cases} \varepsilon\left(\dfrac{K}{\chi}+1\right) = \dfrac{\frac{4}{3}\mu+\mu_v+(\gamma-1)\frac{k}{c_p}}{\rho_0 L c_0}, \\ \dfrac{\varepsilon K(\chi-\lambda+1)}{\chi(\lambda-1)} = \dfrac{k/c_p}{\rho_0 L c_0}. \end{cases} \tag{47}$$

Equations (46) were established under the hypothesis that $\mathfrak{M}$, Q, E are regular functions, but they remain valid if, by passage to the limit, they are applied to the following specific case:

$$\begin{cases} \mathfrak{M}(x_0,x_1) = \mathfrak{M}^*(x_1)\,\delta(x_0), \\ \mathfrak{Q}(x_0,x_1) = \mathfrak{Q}^*(x_1)\,\delta(x_0), \\ E(x_0,x_1) = E^*(x_1)\,\delta(x_0). \end{cases} \tag{48}$$

Here, we must use

$$\left\{\begin{array}{l} \lim \int_{-\infty}^{\infty} E(x_0, x_1)\, dx_0 = E^*(x_1)\,, \\ \lim \int_{-\infty}^{\infty} \tilde{\mathcal{M}}(\lambda^+, \lambda^-)\, d\lambda^\mp = \mathcal{M}^*(\lambda^\pm)\,, \end{array}\right. \tag{49}$$

by means of which eqs.(46) will become

$$\left\{\begin{array}{l} U = \int_{-\infty}^{\infty} \bar{U}_0(x_1 - x_0 - \xi) \dfrac{\exp\{-\xi^2/2\varepsilon(K/\chi+1)x_0\}}{\sqrt{2\pi\varepsilon(K/\chi+1)x_0}}\, d\xi \\ \qquad + \dfrac{1}{\chi}\int_{-\infty}^{\infty} \dfrac{E^*(x_1-\xi)\exp\left\{-\xi^2/4\varepsilon\frac{K(\chi-\lambda+1)}{\chi(\lambda-1)}\right\}}{\sqrt{\frac{4\pi\varepsilon K(\chi-\lambda+1)}{\chi(\lambda-1)}}}\, d\xi \\ V = \int_{-\infty}^{\infty} \bar{V}_0(x_1 + x_0 - \xi) \dfrac{\exp\{-\xi^2/2\varepsilon(\frac{K}{\chi}+1)x_0\}}{\sqrt{2\pi\varepsilon(\frac{K}{\chi}+1)x_0}}\, d\xi \\ \qquad + \dfrac{1}{\chi}\int_{-\infty}^{\infty} \dfrac{E^*(x_1-\xi)\exp\left\{-\xi^2/4\varepsilon\frac{K(\chi-\lambda+1)}{\chi(\lambda-1)}\right\}}{\sqrt{\frac{4\pi\varepsilon K(\chi-\lambda+1)}{\chi(\lambda-1)}x_0}}\, d\xi \\ \Sigma = \int_{-\infty}^{\infty} \dfrac{E^*(x_1-\xi)\exp\left\{-\xi^2/\frac{4\varepsilon K(\chi-\lambda+1)}{\chi(\lambda-1)}x_0\right\}}{\sqrt{\frac{4\pi\varepsilon K(\chi-\lambda+1)}{\chi(\lambda-1)}x_0}}\, d\xi \end{array}\right. \tag{50}$$

provided that we pose

$$\bar{U}_0(x_1) = \mathcal{M}^*(x_1) + \varphi^*(x_1) + \chi^{-1}E^*(x_1)\,, \tag{51}$$

$$\left\{ \quad \bar{V}_0(x_1) = \mathcal{M}^*(x_1) - \mathcal{Q}^*(x_1) + \mathcal{X}^{-1} E^*(x_1) \right.$$

Let us now return to eq.(9) with eq.(48); these expressions are equivalent to the following Cauchy problem:

/V,41

$$\left\{ \begin{aligned} & \frac{\partial \bar{\rho}}{\partial t} + \rho_0 \frac{\partial \bar{u}}{\partial x} = 0, \\ & \rho_0 \frac{\partial \bar{u}}{\partial t} + \frac{\partial \bar{p}}{\partial x} - \left(\frac{4}{3}\mu_0 + \mu_{v_0}\right) \frac{\partial^2 \bar{u}}{\partial x^2} = 0, \\ & \rho_0 T_0 \frac{\partial \bar{S}}{\partial t} - \frac{\partial}{\partial x}\left(k \frac{\partial \bar{T}}{\partial x}\right) = 0, \end{aligned} \right. \tag{52}$$

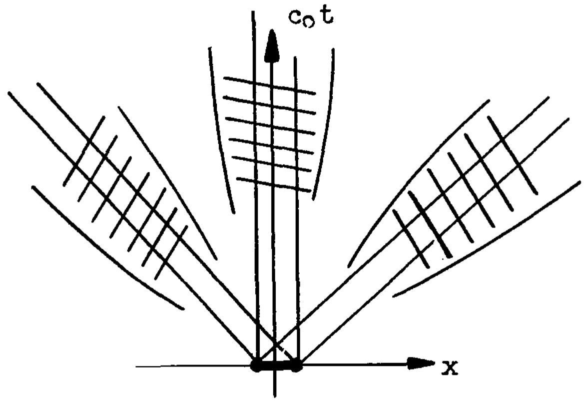
$$\left\{ \begin{aligned} & \bar{u}(0,x) = c_0 \mathcal{M}^*\left(\frac{x}{L}\right), \\ & \bar{\rho}(0,x) = \rho_0 \mathcal{Q}^*\left(\frac{x}{L}\right), \\ & \bar{S}(0,x) = C_v E^*\left(\frac{x}{L}\right), \end{aligned} \right. \tag{53}$$

for which we thus have obtained an approximate solution when the coefficients μ, μ_v, k are small, because of eqs.(50) and (51) and because of

$$\left\{ \begin{aligned} & \bar{u} = \frac{1}{2} c_0 (U - V) \\ & \bar{\rho} = \frac{1}{2} \rho_0 (U + V) \\ & \bar{S} = c_v \Sigma \\ & \bar{T} = \frac{\gamma - 1}{2} T_0 (U + V) + T_0 \Sigma, \end{aligned} \right. \tag{54}$$

if the initial data are zero outside of a small segment, as indicated on the accompanying diagram, while the solution is the superposition of a triple wave system (two acoustic waves, and one entropy wave) which gradually but rapidly extinguishes in domains that open on either side of each geometric wave and whose width increases with time like $\sqrt{t}$. To define this better, let us select the following initial data: /V,42

$$\left\{ \begin{aligned} & \bar{S}(0,x) = 0, \\ & \bar{\rho}(0,x) = \rho_0 c_0^{-1} \bar{u}(0,x) \end{aligned} \right. \tag{55}$$



This will yield

$$V = \Sigma = 0 \tag{56}$$

and, consequently,

$$\begin{aligned} \bar{s}(t, x) &= 0 \\ \bar{\rho}(t, x) &= \rho_0 c_0^{-1} \bar{u}(t, x) \end{aligned} \tag{57}$$

with

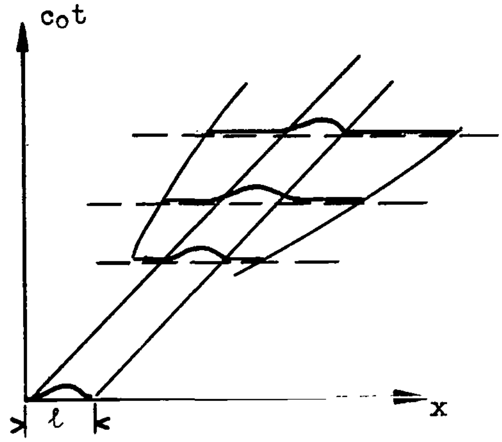
$$\bar{u}(t,x) = \int_{-\infty}^{\infty} \bar{u}(0, x - c_0 t - \xi) \frac{\exp\left\{- \dfrac{\xi^2 \rho_0}{2\left(\frac{4}{3}\mu_0 + \mu_{v_0} + (\gamma-1)\frac{k_0}{c_p}\right) t}\right\}}{\sqrt{\dfrac{2\pi\left\{\frac{4}{3}\mu_0 + \mu_{v_0} + (\gamma-1)\frac{k_0}{c_p}\right\} t}{\rho_0}}} d\xi . \tag{58}$$

When μ_0, μ_{v_0}, k_0 tend to zero, this solution will tend toward the solution of the ideal fluid:

$$\bar{u}_I(t, x) = \bar{u}(0, x - c_0 t) . \tag{59}$$

If the coefficients μ_0, μ_{v_0}, k_0 are only small, then the solution will possess a small signal which delimits the geometric wave and which is exponentially attenuated at a certain distance from the boundaries. Studying now what happens for large values of time, such that

$$\frac{\ell^2 \rho_0}{\mu_0 t} \ll 1 \tag{60}$$



where ℓ denotes the width of the geometric wave, it will be found that eq.(58) differs considerably from eq.(59) and is approximately reduced to /V,43

$$\bar{u}(t,x) \sim \frac{\exp\left\{-\frac{(x-c_0t)^2\rho_0}{2\left(\frac{4}{3}\mu_0+\mu_{v_0}+(\gamma-1)\frac{k_0}{c_p}\right)}\right\}}{\sqrt{\frac{2\pi\left(\frac{4}{3}\mu_0+\mu_{v_0}+(\gamma-1)\frac{k_0}{c_p}\right)t}{\rho_0}}}\int_{-\infty}^{\infty}\bar{u}(0,x)\,dx. \tag{61}$$

The viscosity and thermal conductivity effects thus result in a spreading and an indefinite attenuation of the profile with the passage of time.

If we would have selected

$$\begin{aligned} \bar{s}(0,x) &= 0, \\ \bar{\rho}(0,x) &= -\rho_0 c_0^{-1}\bar{u}(0,x), \end{aligned} \tag{62}$$

we would have obtained

$$\bar{u}(t,x) = \int_{-\infty}^{\infty}\frac{\bar{u}(0,x+c_0t-\xi)\exp\left\{-\frac{\xi^2\rho_0}{2\left\{\frac{4}{3}\mu_0+\mu_{v_0}+(\gamma-1)\frac{k_0}{c_p}\right\}t}\right\}}{\sqrt{\frac{2\pi\left(\frac{4}{3}\mu_0+\mu_{v_0}+(\gamma-1)\frac{k_0}{c_p}\right)t}{\rho_0}}}\,d\xi \tag{63}$$

Finally, with

$$\bar{u}(0,x) = \rho(0,x) = 0, \tag{64}$$

we obtain

$$\bar{u}(t,x) = \bar{\rho}(t,x) = 0 ,$$

$$\bar{T}(t,x) = \int_{-\infty}^{\infty} \frac{\bar{T}(0, x-\xi) \quad \exp\left\{-\frac{\xi^2 \rho_0 c_p}{4k_0 t}\right\}}{\sqrt{\frac{4\pi k_0 t}{\rho_0}}} d\xi . \tag{65}$$

It is of importance to note that the effects described here cannot be obtained directly from eq.(10) by a so-called "naive" expansion process /V,44

$$\begin{cases} U = U_0(x_0,x_1) + \varepsilon U_1(x_0,x_1) + \cdots \\ V = V_0(x_0,x_1) + \varepsilon V_1(x_0,x_1) + \cdots \\ \Sigma = \Sigma_0(x_0,x_1) + \varepsilon \Sigma_1(x_0,x_1) + \cdots , \end{cases} \tag{66}$$

since such a procedure lets the convolution vanish together with the exponentials. According to what we have seen, eq.(66) conveniently describes the phenomenon as long as

$$\varepsilon [x_0] \ll [x_1] \tag{67}$$

denoting by $[x_0]$ and $[x_1]$ the respective scales for x_0 and x_1. Conversely, if the condition (67) ceases to be satisfied, the three waves that describe the phenomenon in first approximation $\{U_0, V_0, \Sigma_0\}$ will have emerged from the initial phase and will have separated, each evolving further on its own. This is indicated by eq.(46); when the spreading and attenuation phenomenon starts playing a more important role, the three modes (two acoustic and one entropy mode) are already separated.

If, in eqs.(50), we set $\epsilon = 0$ we will obtain the following after passage to the limit:

$$\begin{cases} U = \bar{U}_0(x_1 - x_0) - \chi^{-1} E_1^*(x_1) , \\ V = V_0(x_1 + x_0) + \chi^{-1} E^*(x_1) , \\ \Sigma = E^*(x_1) . \end{cases} \tag{68}$$

Let us return to the system (10) and write it in the form of /V,45

$$\begin{cases} \dfrac{\partial U}{\partial x_0} + \dfrac{\partial U}{\partial x_1} + \dfrac{1}{\chi}\dfrac{\partial \Sigma}{\partial x_1} - \epsilon\,\dfrac{\partial^2 U-V}{\partial x_1^2} = \bar{U}_0(x_1)\,\delta(x_0) \\ \dfrac{\partial V}{\partial x_0} - \dfrac{\partial V}{\partial x_1} - \dfrac{1}{\chi}\dfrac{\partial \Sigma}{\partial x_1} - \epsilon\,\dfrac{\partial^2 V-U}{\partial x_1^2} = \bar{V}_0(x_1)\,\delta(x_0) \\ \dfrac{\partial \Sigma}{\partial x_0} - \dfrac{1}{2}\epsilon K\,\dfrac{\partial^2 U+V}{\partial x_1^2} - \dfrac{\epsilon K}{\lambda-1}\dfrac{\partial^2 \Sigma}{\partial x_1^2} = E^*(x_1)\,\delta(x_0) \end{cases} \tag{69}$$

after which, on substitution of eq.(66), we find first

$$\begin{cases} \dfrac{\partial U_0}{\partial x_0} + \dfrac{\partial U_0}{\partial x_1} + \dfrac{1}{\chi}\dfrac{\partial \Sigma_0}{\partial x_1} = \bar{U}_0(x_1)\,\delta(x_0), \\ \dfrac{\partial V_0}{\partial x_0} - \dfrac{\partial V_0}{\partial x_1} - \dfrac{1}{\chi}\dfrac{\partial \Sigma_0}{\partial x_1} = \bar{V}_0(x_1)\,\delta(x_0), \\ \dfrac{\partial \Sigma_0}{\partial x_0} = E^*(x_1)\,\delta(x_0), \end{cases} \tag{70}$$

and then

$$\begin{cases} \dfrac{\partial U_{n+1}}{\partial x_0} + \dfrac{\partial U_{n+1}}{\partial x_1} + \dfrac{1}{\chi}\dfrac{\partial \Sigma_{n+1}}{\partial x_1} = \epsilon\,\dfrac{\partial^2 U_n - V_n}{\partial x_1^2}, \\ \dfrac{\partial V_{n+1}}{\partial x_0} - \dfrac{\partial V_{n+1}}{\partial x_1} - 1/\chi\,\dfrac{\partial \Sigma_{n+1}}{\partial x_1} = \dfrac{\partial^2 V_n - U_n}{\partial x_1^2}, \\ \dfrac{\partial \Sigma_{n+1}}{\partial x_0} = \dfrac{K}{2}\,\dfrac{\partial^2 U_n + V_n}{\partial x_1^2} + \dfrac{K}{\lambda-1}\,\dfrac{\partial^2 \Sigma_n}{\partial x_1^2}, \end{cases} \tag{71}$$

It will be seen that eq.(68) is the solution of eq.(70). Thus, the approximate solution (50) can be obtained from the solution (68) by convolution with the three exponentials which, as indicated in the specific case of eq.(61), can be written in such a manner that the sound is attenuated and dispersed by dissipation. It would be highly instructive to continue the expansion that had yielded eq.(45) and to derive from this that eq.(66) is almost correct under the conditions of eq.(67), so that eq.(66) can be corrected by convolution with the same exponentials. /V,46

5.2.2 Phenomena of Nonlinear Convection

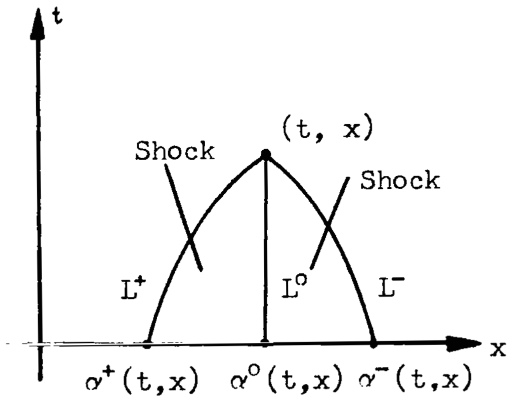
We will now neglect the viscosity and conductivity effects and, instead, take into consideration the effects of nonlinear convection. For this, we use the equations

$$\left\{\begin{aligned} &\frac{\partial p^+}{\partial t} + (u+c)\frac{\partial p^+}{\partial x} + \left(\tau g_s - c\frac{\partial p^+}{\partial s}\right)\frac{\partial s}{\partial x} = 0, \\ &\frac{\partial p^-}{\partial t} + (u-c)\frac{\partial p^-}{\partial x} - \left(\tau g_s - c\frac{\partial p^-}{\partial s}\right)\frac{\partial s}{\partial x} = 0, \\ &\frac{\partial s}{\partial t} + u\frac{\partial s}{\partial x} = 0, \end{aligned}\right. \tag{72}$$

to which we associate initial conditions, for example,

$$\left\{\begin{aligned} &p^+(0,x) = \varepsilon c_0 F(x), \\ &p^-(0,x) = \varepsilon c_0 G(x), \\ &s(0,x) = s_0, \end{aligned}\right. \qquad \varepsilon \ll 1 \tag{73}$$

as well as Rankine-Hugoniot conditions for the shocks.



As the solution, for a given point of the plane (t, x) let us pass (toward decreasing t) the three integral curves of the equations

$$\frac{dx}{dt} = u + \varepsilon c \qquad \varepsilon = 0, \pm 1 \tag{73a}$$

and let us denote these curves by L^0, L^+, and L^-; we can then give the /V,47

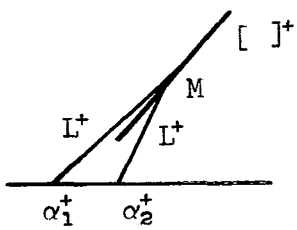
following form to eq.(72):

$$\left\{\begin{aligned} P^+ &= \varepsilon c_0 F(\alpha^+) + \int_{L^+}\left(\tau g_s - c\frac{\partial P^+}{\partial s}\right)\frac{\partial s}{\partial x}\,dt + \sum_{L^+}[P^+]^- + \sum_{L^+}[P^+]^0 \\ P^- &= \varepsilon c_0 G(\alpha^-) - \int_{L^-}\left(\tau g_s - c\frac{\partial P^-}{\partial s}\right)\frac{\partial s}{\partial x}\,dt + \sum_{L^-}[P^-]^+ + \sum_{L^-}[P^-]^0 \\ S &= \sum_{L^0}[S]^+ + \sum_{L^0}[S]^- + S_0 , \end{aligned}\right. \tag{74}$$

with the notations α^0, α^+, and α^- for the abscissas of the points where the curves L^0, L^+, and L^- intersect the axis $t = 0$. On passage of a shock, we have

$$[f] = f(t+0, x) - f(t-0, x) \tag{75}$$

which is the discontinuity of f. If the perturbations are weak, as is the case here, the shocks are close to the characteristics; we denote by $[\ ]^+$ the discontinuity across a shock close to $\frac{dx}{dt} = + c_0$, while $[\ ]^-$ is the discontinuity on passage of a shock close to $\frac{dx}{dt} = - c_0$. The sum $\sum_{L^+}$ refers to shocks that traverse L^+ while the sum $\sum_{L^-}$ refers to shocks that traverse L^-; the same is true for $\sum_{L^0}$. The curves L^+ and L^- can also traverse contact discontinuities that are produced by interaction of the shocks; this leads us to denote by $[f]^0$



the corresponding discontinuity of f. In writing eq.(74), we admitted that L^+ traversed no shock $[\ ]^+$ and that L^- traversed no shock $[\ ]^-$. In fact, if /V,48 we plot an integral curve of $\frac{dx}{dt} = u + c$ from $t = 0$, $x = \alpha^+$, we will have to terminate this curve as soon as it encounters a shock $[\ ]^+$. Through the shock point M, two such curves are passing, emerging respectively from $x = \alpha_1^+$ and α_2^+ on $t = 0$, while in the vicinity of the shock itself, the points to the right

are on L^+ with $\alpha^+ > \alpha_2^+$ while the points to the left are on L^+ with $\alpha^+ < \alpha_1^+$. Thus, on progressing in the direction of increasing t, any L^+ must be stopped as soon as it encounters a shock $[\ ]^+$ and any L^- as soon as it encounters a shock $[\ ]^-$. Naturally, in writing eq.(74), we must proceed in the opposite direction, i.e., in the sense of decreasing t, making it impossible to rigorously prove a priori that any point (t, x) can be joined to the axis t = 0 by a segment of L^+ and a segment of L^-, stopped on encounter with a shock of the same type. It is exactly this point, among others, which makes the difficulties of the existence theory in flows with shock almost insurmountable.

Before making use of eq.(74) for obtaining an approximating solution of the problem, we must briefly discuss the classical study (see the Course on Gas Dynamics) of shocks. We systematically will use the following notation

$$[A] = A_2 - A_1 \qquad \widehat{A} = A_2 + A_1 \tag{76}$$

and recall the relations

/V,49

$$\left\{\begin{aligned} [AB] &= \tfrac{1}{2}\left([A]\widehat{B} + \widehat{A}[B]\right), \\ [A^2] &= [A]\widehat{A}, \\ [A^3] &= \tfrac{1}{2}[A]\left((\widehat{A})^2 + \widehat{A^2}\right) = \tfrac{1}{4}[A]\left(3(\widehat{A})^2 + [A]^2\right), \\ \widehat{AB} &= \tfrac{1}{2}\left(\widehat{A}\,\widehat{B} + [A][B]\right), \\ \widehat{A^2} &= \tfrac{1}{2}\left((\widehat{A})^2 + [A]^2\right). \end{aligned}\right. \tag{77}$$

Let τ_0, S_0 be the reference state and let us use the following Taylor series:

$$\left\{\begin{aligned} p &= p_0 + g_{\tau_0}(\tau-\tau_0) + \tfrac{1}{2} g_{\tau_0\tau_0}(\tau-\tau_0)^2 + \tfrac{1}{6} g_{\tau_0\tau_0\tau_0}(\tau-\tau_0)^3 + \cdots \\ &\qquad + g_{S_0}(S-S_0) + \cdots \\ h &= e + p\tau = h_0 + \tau_0 g_{\tau_0}(\tau-\tau_0) + \tfrac{1}{2}\left(\tau_0 g_{\tau_0\tau_0} + g_{\tau_0}\right)(\tau-\tau_0)^2 + \\ &\qquad + \tfrac{1}{6}\left(\tau_0 g_{\tau_0\tau_0\tau_0} + 2 g_{\tau_0\tau_0}\right)(\tau-\tau_0)^3 + \left(\tau_0 g_{S_0} + T_0\right)(S-S_0) + \cdots \end{aligned}\right. \tag{78}$$

We leave to the reader the task of deriving this expression by making use of the following relation:

$$de = -p\,d\tau + T\,ds \tag{79}$$

We will modify the writing of this equation by stating that one can also use

$$\left\{\begin{aligned} g_\tau &= -\frac{c^2}{\tau^2}, \\ g_{\tau\tau} &= 2\Gamma\frac{c^2}{\tau^3}, \\ g_{\tau\tau\tau} &= -2\left\{\Lambda + \Gamma(2\Gamma+1)\right\}\frac{c^2}{\tau^4}, \end{aligned}\right. \tag{80}$$

where Γ has already been defined $\left(\tau\frac{c_T}{c} = 1 - \Gamma\right)$ and where

$$\Lambda = -\tau\,\Gamma_\tau. \tag{81}$$

Let us note that, for an ideal gas, Γ is a constant and that $\Lambda = 0$. This will yield /V,50

$$\left\{\begin{aligned} p &= p_0 - \frac{c_0^2}{\tau_0}\frac{\tau-\tau_0}{\tau_0} + \Gamma_0\frac{c_0^2}{\tau_0}\left(\frac{\tau-\tau_0}{\tau_0}\right)^2 - \frac{1}{3}\left\{\Lambda_0 + \Gamma_0(2\Gamma_0+1)\right\}\frac{c_0^2}{\tau_0}\left(\frac{\tau-\tau_0}{\tau_0}\right)^3 + \\ &\qquad + g_{s_0}(s-s_0) + \cdots, \\ h &= h_0 - c_0^2\frac{\tau-\tau_0}{\tau_0} + \frac{1}{2}(2\Gamma_0-1)c_0^2\left(\frac{\tau-\tau_0}{\tau_0}\right)^2 - \\ &\qquad - \frac{1}{3}\left\{\Lambda_0 + \Gamma_0(2\Gamma_0-1)\right\}c_0^2\left(\frac{\tau-\tau_0}{\tau_0}\right)^3 + (\tau_0 g_{s_0} + T_0)(s-s_0) + \cdots. \end{aligned}\right. \tag{82}$$

In a similar manner, the reader will obtain

$$\left\{\begin{aligned} P^+ &= u - c_0\frac{\tau-\tau_0}{\tau_0}\left\{1 - \frac{\Gamma_0}{2}\frac{\tau-\tau_0}{\tau_0} + \frac{\Lambda_0 + \Gamma_0(\Gamma_0+1)}{6}\left(\frac{\tau-\tau_0}{\tau_0}\right)^2 + \cdots\right\} \\ P^- &= -u - \frac{c_0}{\tau_0}(\tau-\tau_0)\left\{1 - \frac{\Gamma_0}{2}\frac{\tau-\tau_0}{\tau_0} + \frac{\Lambda_0 + \Gamma_0(\Gamma_0+1)}{6}\left(\frac{\tau-\tau_0}{\tau_0}\right)^2 + \cdots\right\} \end{aligned}\right. \tag{83}$$

from which one can derive

$$u = \frac{1}{2}\left(P^+ - P^-\right) \tag{84}$$

$$\left\{ \frac{\tau-\tau_0}{\tau_0} = -\frac{1}{2}\,\frac{p^+ + p^-}{c_0}\left\{1 - \frac{\Gamma_0}{4}\,\frac{p^+ + p^-}{c_0} + \frac{\Lambda_0+\Gamma_0-2\Gamma_0^2}{24}\left(\frac{p^+ + p^-}{c_0}\right)^2 + O\left(\frac{p^+ + p^-}{c_0}\right)^3 + O\left(\frac{S-S_0}{S_0}\right)\right\} \right.$$

and, finally

$$\left\{ \begin{aligned} u+c &= c_0 + \frac{1}{2}(p^+ - p^-) + \frac{\Gamma_0-1}{2}(p^+ + p^-) + \frac{\Lambda_0}{8}c_0\left(\frac{p^+ + p^-}{c_0}\right)^2 \\ &\quad + c_0\,O\left(\frac{S-S_0}{S_0}\right) + c_0\,O\left(\frac{p^+ + p^-}{c_0}\right)^3, \\ u-c &= -c_0 + \frac{1}{2}(p^+ - p^-) - \frac{\Gamma_0-1}{2}(p^+ + p^-) - \frac{\Lambda_0}{8}c_0\left(\frac{p^+ + p^-}{c_0}\right)^2 \\ &\quad + c_0\,O\left(\frac{S-S_0}{S_0}\right) + c_0\,O\left(\frac{p^+ + p^-}{c_0}\right)^3, \end{aligned} \right. \tag{85}$$

which is obtained because of /V,51

$$c_\tau = \frac{(1-\Gamma)c}{\tau}, \qquad c_{\tau\tau} = \frac{\Lambda + \Gamma(\Gamma-1)}{\tau^2}c, \tag{86}$$

$$\int_{\tau_0}^{\tau}\frac{c}{\tau}\,d\tau = \frac{c_0}{\tau_0}\int_{\tau_0}^{\tau}\left\{1 - \Gamma_0\,\frac{\tau_1-\tau_0}{\tau_0} + \frac{\Lambda_0+\Gamma_0(\Gamma_0+1)}{2}\left(\frac{\tau_1-\tau_0}{\tau_0}\right)^2 + \cdots\right\}d\tau_1. \tag{87}$$

Let us now consider a shock whose velocity of displacement is

$$W = \frac{dx_c}{dt}, \tag{88}$$

so that the shock conditions are written as follows: A scalar m exists such that

$$\left\{ \begin{aligned} &2W = \bar{u} + m\,\bar{\tau}, && a) \\ &[u] - [\tau]m = 0, && b) \\ &[p] + m^2[\tau] = 0, && c) \end{aligned} \right. \tag{89}$$

$$\left(\quad [h] + \frac{m^2}{2}[\tau^2] = 0 . \right. \qquad d)$$

From these conditions, we can derive the Hugoniot relation

$$[h] - \frac{1}{2}[p]\,\overline{\tau} = 0 \tag{90}$$

However, by an expansion in Taylor series to the vicinity of $\tau = \tau_0$, $S = S_0$, we obtain

$$[h] - \frac{1}{2}[p]\,\overline{\tau} \equiv \frac{1}{12}\, g_{\tau_0\tau_0}[\tau-\tau_0]\left\{2\,\overline{(\tau-\tau_0)^2} - \left(\overline{\tau-\tau_0}\right)^2\right\} + T_0[S-S_0] + \dots$$

from which it follows that

$$[S-S_0] = -\frac{g_{\tau_0\tau_0}}{12\,T_0}[\tau-\tau_0]^3 , \tag{91}$$

a result that had already been used above. The reader, as exercise, can then prove that we have

/V,52

$$[S-S_0] = -\frac{1}{24}\frac{\overline{g_{\tau\tau}}}{\overline{T}}[\tau-\tau_0]^3 + O\left\{S_0\left[\frac{\tau-\tau_0}{\tau_0}\right]^5\right\} . \tag{92}$$

From eq.(89c) we derive

$$m^2 = -\frac{[p]}{[\tau]} = \frac{c_0^2}{\tau_0}\left\{1 - \Gamma_0\,\overline{\frac{\tau-\tau_0}{\tau_0}} + \frac{\Lambda_0 + \Gamma_0(2\Gamma_0+1)}{4}\left(\overline{\frac{\tau-\tau_0}{\tau_0}}\right)^2 + \right.$$
$$\left. + \left(\frac{\Lambda_0 + \Gamma_0(2\Gamma_0+1)}{12} + \frac{\Gamma_0}{6}\frac{\tau_0 g_{S_0}}{T_0}\right)\left[\frac{\tau-\tau_0}{\tau_0}\right]^2 + \dots\right\} , \tag{93}$$

and, consequently, for the velocity of the shock

$$W^{\pm} = \frac{1}{2}\overline{u} \pm c_0\left\{1 + \frac{1-\Gamma_0}{2}\,\overline{\frac{\tau-\tau_0}{\tau_0}} + \frac{\Lambda_0 + (\Gamma_0-1)\Gamma_0}{8}\left(\overline{\frac{\tau-\tau_0}{\tau_0}}\right)^2 + \right.$$
$$\left. + \left(\frac{\Lambda_0 + \Gamma_0(2\Gamma_0+1)}{24} + \frac{\Gamma_0}{12}\frac{\tau_0 g_{S_0}}{T_0}\right)\left[\frac{\tau-\tau_0}{\tau_0}\right]^2 + \dots\right\} , \tag{94}$$

which is transformed because of eq.(84)

$$W^{\pm} = \frac{1}{4}\,\overline{p^{+} + p^{-}} \pm c_0 \left\{ 1 + \frac{\Gamma_0 - 1}{4}\,\frac{\overline{p^{+} + p^{-}}}{c_0} + \frac{\Lambda_0 - \Gamma_0(\Gamma_0 - 1)}{32}\left(\frac{\overline{p^{+} + p^{-}}}{c_0}\right)^2 + \left(\frac{\Lambda_0 + \Gamma_0(2\Gamma_0 + 1)}{96} + \frac{\Gamma_0}{48}\,\frac{\tau_0 g_{s0}}{\rho_0}\right)\left[\frac{p^{+} + p^{-}}{c_0}\right]^2 + \cdots \right\} \tag{95}$$

However, according to eq.(85), we have

$$\frac{1}{2}\,\overline{u \pm c} = \frac{1}{4}\,\overline{p^{+} + p^{-}} \pm c_0 \left\{ 1 + \frac{\Gamma_0 - 1}{4}\,\frac{\overline{p^{+} + p^{-}}}{c_0} + \frac{\Lambda_0}{32}\left(\frac{\overline{p^{+} + p^{-}}}{c_0}\right)^2 + \frac{\Lambda_0}{32}\left[\frac{p^{+} + p^{-}}{c_0}\right]^2 + \cdots \right\}, \tag{96}$$

such that

/V.53

$$W^{\pm} = \frac{1}{2}\,\overline{u \pm c} \pm c_0 \left\{ \frac{\Gamma_0(1 - \Gamma_0)}{32}\left(\frac{\overline{p^{+} + p^{-}}}{c_0}\right)^2 + \left(\frac{\Gamma_0(2\Gamma_0 + 1) - 2\Lambda_0}{96} + \frac{\Gamma_0}{48}\,\frac{\tau_0 g_{s0}}{\rho_0}\right)\left[\frac{p^{+} + p^{-}}{c_0}\right]^2 + \cdots \right\}. \tag{97}$$

Consequently, we can formulate the following statement: The velocity of a shock, neglecting the squares of the perturbations and their predicted products, is equal to the arithmetic mean of the velocities of acoustic waves that end there on either side. This property is not conserved when passing to a higher approximation order. We then can use eq.(89b); this yields a direct result since the ± signs correspond (depending on whether they are in the upper or lower position) to those of eq.(95); this yields successively

$$\left\{\begin{aligned}
&\left[\frac{\tau-\tau_0}{\tau_0}\right] = -\frac{1}{2}\left[\frac{p^+ + p^-}{c_0}\right]\Bigg\{1 - \frac{\Gamma_0}{4}\,\frac{\overline{p^+ + p^-}}{c_0} + \frac{2\Gamma_0^2 - \Gamma_0 - \Lambda_0}{32}\left(\frac{\overline{p^+ + p^-}}{c_0}\right)^2 + \\
&\qquad + \frac{2\Gamma_0^2 - \Gamma_0 - \Lambda_0}{96}\left[\frac{p^+ + p^-}{c_0}\right]^2 + \cdots\Bigg\} \\
&\tau_0 m = \mp c_0\Bigg\{1 + \frac{\Gamma_0}{4}\,\frac{\overline{p^+ + p^-}}{c_0} + \frac{\Lambda_0 + \Gamma_0}{32}\left(\frac{\overline{p^+ + p^-}}{c_0}\right)^2 + \\
&\qquad + \left(\frac{\Lambda_0 + \Gamma_0(1-\Gamma_0)}{96} + \frac{\Gamma_0 \tau_0 g_{S_0}}{48 T_0}\right)\left[\frac{p^+ + p^-}{c_0}\right]^2 + \cdots\Bigg\} \\
&[p^+ - p^-] \mp [p^+ + p^-]\left\{1 + \left(\frac{\Gamma_0^2}{96} + \frac{\Gamma_0 \tau_0 g_{S_0}}{48 T_0}\right)\left[\frac{p^+ + p^-}{c_0}\right]^2 + \cdots\right\} = 0\,.
\end{aligned}\right. \tag{98}$$

Theorem 3: Let us assume that the gas is slightly perturbed, starting from a reference state

/V,54

$$\tau = \tau_0\,, \qquad S = S_0\,, \qquad u = 0 \tag{99}$$

The perturbations can be represented by means of the variables

$$p^+ = u - \int_{\tau_0}^{\tau} \frac{c(\tau_1, S)}{\tau_1}\,d\tau_1\,, \qquad p^- = -u - \int_{\tau_0}^{\tau} \frac{c(\tau_1, S)}{\tau_1}\,d\tau_1\,, \qquad S \tag{100}$$

yielding

$$\left\{\begin{aligned}
&u = \frac{1}{2}(p^+ - p^-)\,, \\
&\frac{\tau-\tau_0}{\tau_0} = -\frac{1}{2}\,\frac{p^+ + p^-}{c_0}\Bigg\{1 - \frac{\Gamma_0}{4}\,\frac{p^+ + p^-}{c_0} + \frac{2\Gamma_0^2 - \Gamma_0 - \Lambda_0}{24}\left(\frac{p^+ + p^-}{c_0}\right)^2 + \\
&\qquad + O\left(\left(\frac{p^+ + p^-}{c_0}\right)^3\right) + O\left(\frac{S - S_0}{S_0}\right)\Bigg\}
\end{aligned}\right. \tag{101}$$

Two families of possible shocks exist whose propagation velocity is as follows:

$$
\left\{
\begin{aligned}
W^{+} &= c_0 + \frac{1}{4}\,\overline{p^{+}-p^{-}}' + \frac{\Gamma_0-1}{4}\,\overline{p^{+}+p^{-}}' + \\
&\quad + \frac{\Lambda_0-\Gamma_0(\Gamma_0-1)}{32}\,c_0\left(\frac{\overline{p^{+}+p^{-}}}{c_0}\right)^2 + \\
&\quad + \left(\frac{\Lambda_0+\Gamma_0(2\Gamma_0+1)}{96} + \frac{\Gamma_0}{48}\,\frac{\tau_0 g_{s_0}}{T_0}\right)\left[\frac{p^{+}+p^{-}}{c_0}\right]^2 + \cdots \qquad (C^{+}) \\
W^{-} &= -c_0 + \frac{1}{4}\,\overline{p^{+}-p^{-}}' - \frac{\Gamma_0-1}{4}\,\overline{p^{+}+p^{-}}' - \\
&\quad - \frac{\Lambda_0-\Gamma_0(\Gamma_0-1)}{32}\,c_0\left(\frac{\overline{p^{+}+p^{-}}}{c_0}\right)^2 - \qquad (C^{-}) \\
&\quad - \left(\frac{\Lambda_0+\Gamma_0(2\Gamma_0+1)}{96} + \frac{\Gamma_0}{48}\,\frac{\tau_0 g_{s_0}}{T_0}\right)\left[\frac{p^{+}+p^{-}}{c_0}\right]^2 + \cdots ,
\end{aligned}
\right. \qquad (102)
$$

where [f] denotes the discontinuity of f while $\frac{1}{2}\,\overline{f}$ represents the mean of the values of f on either side of the shock. On passage of a shock (C^{+}), we have

/V,55

$$
\left\{
\begin{aligned}
[p^{-}]^{+} &= -c_0\left(\frac{\Gamma_0^2}{192} + \frac{\Gamma_0\,\tau_0 g_{s_0}}{96\,T_0}\right)\left(\left[\frac{p^{+}}{c_0}\right]^{+}\right)^3 + \cdots , \\
T_0[s-s_0]^{+} &= \frac{\Gamma_0\,c_0^2}{48}\left(\left[\frac{p^{+}}{c_0}\right]^{+}\right)^3 + \cdots ,
\end{aligned}
\right. \qquad (C^{+}) \qquad (103)
$$

whereas, on passage of a shock (C^{-}), we have

$$
\left\{
\begin{aligned}
[p^{+}]^{-} &= -c_0\left(\frac{\Gamma_0^2}{192} + \frac{\Gamma_0\,\tau_0 g_{s_0}}{96\,T_0}\right)\left(\left[\frac{p^{-}}{c_0}\right]^{-}\right)^3 + \cdots , \\
T_0[s-s_0]^{-} &= \frac{\Gamma_0\,c_0^2}{48}\left(\left[\frac{p^{-}}{c_0}\right]^{-}\right)^3 + \cdots .
\end{aligned}
\right. \qquad (C^{-}) \qquad (104)
$$

Let us now return to eqs.(74) which show that

$$P^{+} = c_0\, O(\varepsilon) \quad , \qquad P^{-} = c_0\, O(\varepsilon) \tag{105}$$

so that, because of eqs.(103) and (104), we have

$$\left\{ \begin{array}{ll} [P^{-}]^{+} = c_0\, O(\varepsilon^3) & [P^{+}]^{-} = c_0\, O(\varepsilon^3) \\ [S-S_0]^{+} = S_0\, O(\varepsilon^3) & [S-S_0]^{-} = S_0\, O(\varepsilon^3) \end{array} \right. \tag{106}$$

whence

$$S = S_0 + S_0\, O(\varepsilon^3) \tag{107}$$

after which eqs.(74) can be written as follows:

$$\left\{ \begin{array}{l} P^{+} = \varepsilon c_0\, F(\alpha^{+}) \left\{ 1 + O(\varepsilon^2) \right\} , \\ P^{-} = \varepsilon c_0\, G(\alpha^{-}) \left\{ 1 + O(\varepsilon^2) \right\} , \\ S = S_0 \left\{ 1 + O(\varepsilon^3) \right\} . \end{array} \right. \tag{108}$$

To express the solution of eqs.(72), (73), to within $O(\varepsilon^2)$, it merely is necessary to define t and x as a function of α^{+} and α^{-}. To obtain this, we will /V,56 integrate the following equations:

$$\frac{\partial x}{\partial \alpha^{-}} = (u+c) \frac{\partial t}{\partial \alpha^{-}} \quad , \qquad \frac{\partial x}{\partial \alpha^{+}} = (u-c) \frac{\partial t}{\partial \alpha^{+}} \tag{109}$$

which we transform by posing

$$x - c_0 t = \xi \quad , \qquad x + c_0 t = \eta , \tag{110}$$

in such a manner as to obtain

$$\left\{ \begin{array}{l} \dfrac{\partial \xi}{\partial \alpha^{-}} = \dfrac{u+c-c_0}{2c_0} \dfrac{\partial \eta - \xi}{\partial \alpha^{-}} , \\ \dfrac{\partial \eta}{\partial \alpha^{+}} = - \dfrac{u - (c-c_0)}{2c_0} \dfrac{\partial \xi - \eta}{\partial \alpha^{+}} . \end{array} \right. \tag{111}$$

However, we have

$$\frac{u \pm c - c_0}{c_0} = \frac{1}{2} \frac{P^{+} - P^{-}}{c_0} \pm \frac{\Gamma_0 - 1}{2} \frac{P^{+} + P^{-}}{c_0} \pm \frac{\Lambda_0}{8} \left(\frac{P^{+} + P^{-}}{c_0} \right)^2 + O(\varepsilon^3) , \tag{112}$$

such that eq.(111) can be written as

$$\left\{\begin{aligned} \frac{\partial \xi}{\partial \alpha^-} &= \frac{\varepsilon}{4}\left\{ \Gamma_0\, F(\alpha^+) + (\Gamma_0 - 2)\, G(\alpha^-) + \varepsilon \frac{\Lambda_0}{4}\left(F(\alpha^+) + G(\alpha^-)\right)^2 + \right. \\ &\qquad \left. + \, O(\varepsilon^2) \right\} \frac{\partial\, \eta - \xi}{\partial \alpha^-} \\ \frac{\partial \eta}{\partial \alpha^+} &= \frac{\varepsilon}{4}\left\{ \Gamma_0\, G(\alpha^-) + (\Gamma_0 - 2)\, F(\alpha^+) + \varepsilon \frac{\Lambda_0}{4}\left(F(\alpha^+) + G(\alpha^-)\right)^2 + \right. \\ &\qquad \left. + \, O(\varepsilon^2) \right\} \frac{\partial\, \xi - \eta}{\partial \alpha^+} \end{aligned}\right. \tag{113}$$

whose solution can be looked for in the following form:

$$\begin{aligned} \xi &= \xi_0(\alpha^+, \alpha^-) + \varepsilon\, \xi_1(\alpha^+, \alpha^-) + \varepsilon^2 \xi_2(\alpha^+, \alpha^-) + \cdots \\ \eta &= \eta_0(\alpha^+, \alpha^-) + \varepsilon\, \eta_1(\alpha^+, \alpha^-) + \varepsilon^2 \eta_2(\alpha^+, \alpha^-) + \cdots . \end{aligned} \tag{114}$$

Substituting eq.(114) into eq.(113) will yield /V,57

$$\frac{\partial \xi_0}{\partial \alpha^-} = 0 , \qquad \frac{\partial \eta_0}{\partial \alpha^+} = 0 , \tag{115}$$

$$\left\{\begin{aligned} \frac{\partial \xi_1}{\partial \alpha^-} &= \left\{ \frac{\Gamma_0}{4} F(\alpha^+) + \frac{\Gamma_0 - 2}{4} G(\alpha^-) \right\} \frac{\partial\, \eta_0 - \xi_0}{\partial \alpha^-} , \\ \frac{\partial \eta_1}{\partial \alpha^+} &= \left\{ \frac{\Gamma_0}{4} G(\alpha^-) + \frac{\Gamma_0 - 2}{4} F(\alpha^+) \right\} \frac{\partial\, \xi_0 - \eta_0}{\partial \alpha^+} , \end{aligned}\right. \tag{116}$$

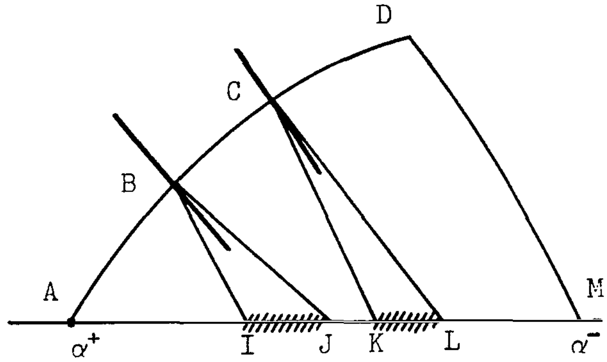
$$\left\{\begin{aligned} \frac{\partial \xi_2}{\partial \alpha^-} &= \left\{ \frac{\Gamma_0}{4} F(\alpha^+) + \frac{\Gamma_0 - 2}{4} G(\alpha^-) \right\} \frac{\partial\, \eta_1 - \xi_1}{\partial \alpha^-} + \frac{\Lambda_0}{16}\left(F(\alpha^+) + G(\alpha^-)\right)^2 \frac{\partial\, \eta_0 - \xi_0}{\partial \alpha^-} , \\ \frac{\partial \eta_2}{\partial \alpha^+} &= \left\{ \frac{\Gamma_0}{4} G(\alpha^-) + \frac{\Gamma_0 - 2}{4} F(\alpha^+) \right\} \frac{\partial\, \xi_1 - \eta_1}{\partial \alpha^+} + \frac{\Lambda_0}{16}\left(F(\alpha^+) + G(\alpha^-)\right)^2 \frac{\partial\, \xi_0 - \eta_0}{\partial \alpha^+} , \end{aligned}\right. \tag{117}$$

resulting in the initial conditions

$$\xi_0(\alpha^+, \alpha^+) = \alpha^+ , \qquad \eta_0(\alpha^-, \alpha^-) = \alpha^- , \tag{118}$$

$$\left\{ \quad \xi_n(\alpha^+,\alpha^+) = 0, \qquad \eta_m(\alpha^-,\alpha^-) = 0, \qquad n \geqslant 1. \right.$$

However, it should be remembered that eqs.(116) and (117) as well as the following equations are applicable only between shocks. For example, in the case of the accompanying diagram, the characteristic ABCD encounters two shocks (C^{-1}),



so that eqs.(116a) and (117a) are no longer applicable along the hatched segments.

For example, when the integration of eq.(115) is trivial, that of eq.(116) will yield

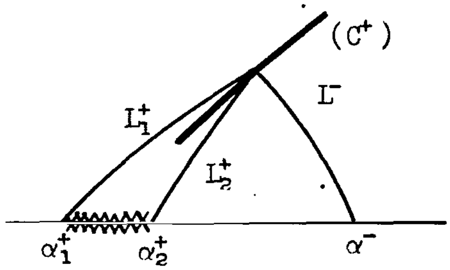
$$\left\{ \begin{aligned} \xi_1 &= \frac{\Gamma_0}{4} F(\alpha^+)\, \Delta^+(\alpha_+,\alpha^-) + \frac{\Gamma_0 - 2}{4} \int_{\Delta^+(\alpha^+,\alpha^-)} G(z)\, dz \\ \eta_1 &= \frac{\Gamma_0}{4} G(\alpha^-)\, \Delta^-(\alpha^-,\alpha^+) + \frac{\Gamma_0 - 2}{4} \int_{\Delta^-(\alpha^-,\alpha^+)} F(z)\, dz , \end{aligned} \right. \tag{119}$$

denoting by $\Delta^+(\alpha_+, \alpha_-)$ the interval (α^+, α^-) from which the segments inter- /V,58 cepted by the shocks (C^-) are excluded; similarly, $\Delta^-(\alpha^-, \alpha^+)$ denotes the interval (α^-, α^+) from which the segments intercepted by the shocks (C^+) are excluded. For ξ_2 and η_2, we have a formulation which is similar in principle. Neglecting $O(\epsilon^2)$, we obtain

$$\left\{ \begin{aligned} \xi &= \alpha^+ + \varepsilon \left\{ \frac{\Gamma_0}{4} F(\alpha^+)\, \Delta^+(\alpha^+,\alpha^-) + \frac{\Gamma_0 - 2}{4} \int_{\Delta^+(\alpha^+,\alpha^-)} G(z)\, dz \right\} + O(\varepsilon^2), \\ \eta &= \alpha^- + \varepsilon \left\{ \frac{\Gamma_0}{4} G(\alpha^-)\, \Delta^-(\alpha^-,\alpha^+) + \frac{\Gamma_0 - 2}{4} \int_{\Delta^-(\alpha^-,\alpha^+)} F(z)\, dz \right\} + O(\varepsilon^2). \end{aligned} \right. \tag{120}$$

On traversing a shock (C^+), the quantities α^-, $G(\alpha^-)$, ξ, η will be continuous

whereas α^+ and $F(\alpha^+)$ will be discontinuous. Let us consider the case of the accompanying diagram, where α^+ passes from α_1^+ to α_2^+ while $\Delta^+(\alpha^+, \alpha^-)$ passes



from $\alpha^- - \alpha_1^+$ to $\alpha^- - \alpha_2^+$; if L^+ traverses shocks of the C^- type, nothing will be changed in this respect. Let us write the continuity of ξ, yielding

$$\alpha_2^+ - \alpha_1^+ + \varepsilon\left\{\frac{\Gamma_0}{4}\Big(F(\alpha_2^+)(\alpha^- - \alpha_2^+) - F(\alpha_1^+)(\alpha^- - \alpha_1^+)\Big) + \frac{\Gamma_0 - 2}{4}\int_{\alpha_1^+}^{\alpha_2^+} G(z)\, dz\right\} + O(\varepsilon^2) = 0, \tag{121}$$

while the continuity of η yields

$$\Gamma_0\,(\alpha_2^+ - \alpha_1^+) + (\Gamma_0 - 2)\int_{\alpha_1^+}^{\alpha_2^+} F(z)\, dz + O(\varepsilon) = 0. \tag{122}$$

The first condition shows that $[\alpha^+] = O(\varepsilon)$, and the second condition contributes nothing to the order considered here. Taking into consideration that $[\alpha^+] = O(\varepsilon)$, the first condition can be written as /V,59

$$[\alpha^+] + \varepsilon\,\frac{\Gamma_0}{4}\left(\alpha^- - \frac{\overline{\alpha^+}}{2}\right)\left[F(\alpha^+)\right] = 0, \tag{127}$$

which furnishes a first correlation between the values α_1^+ and α_2^+ on either side of the shock. To obtain another relation, it is necessary to write that the velocity of the shock is given by

$$W^+ = c_0 + \varepsilon\left\{\frac{\Gamma_0 - 2}{2}G(\alpha^-) + \frac{\Gamma_0}{4}\,\overline{F(\alpha^+)}\right\}c_0. \tag{128}$$

Before attempting to utilize this relation, let us mention in passing that, a posteriori, one can state that it is not necessary, in $\alpha^- - \frac{1}{2}\overline{\alpha^+}$, to make allowance for the sum of the segment lengths intercepted by the shocks (C^-), since this sum is $O(\epsilon)$ so that the corresponding correction belongs to the order of $O(\epsilon^2)$. For the same reason, we can write

$$\left\{\begin{aligned} \xi &= \alpha^+ + \varepsilon\left\{\frac{\Gamma_0}{4}(\alpha^- - \alpha^+)F(\alpha^+) + \frac{\Gamma_0 - 2}{4}\int_{\alpha^+}^{\alpha^-} G(\tau)\,d\tau\right\} + O(\varepsilon^2) \\ \eta &= \alpha^- + \varepsilon\left\{\frac{\Gamma_0}{4}(\alpha^+ - \alpha^-)G(\alpha^-) + \frac{\Gamma_0 - 2}{4}\int_{\alpha^-}^{\alpha^+} F(\tau)\,d\tau\right\} + O(\varepsilon^2) \end{aligned}\right. \tag{129}$$

instead of eq.(120). Let us return to eq.(128), from which we derive that, along a shock (C^+), we have

$$\frac{d\xi}{d\eta} = \frac{W^+/c_0 - 1}{W^+/c_0 + 1} = \varepsilon\left(\frac{\Gamma_0 - 2}{4}G(\alpha^-) + \frac{\Gamma_0}{8}\overline{F(\alpha^+)}\right) + o(\varepsilon^2) \tag{130}$$

from which it follows, on each face of the shock, that

$$\begin{aligned} d\alpha^+ + \varepsilon\Big\{&\frac{\Gamma_0}{4}F(\alpha^+)(d\alpha^- - d\alpha^+) + \frac{\Gamma_0}{4}(\alpha^- - \alpha^+)F'(\alpha^+)\,d\alpha^+ + \\ &+ \frac{\Gamma_0 - 2}{4}\left(G(\alpha^-)d\alpha^- - G(\alpha^+)\,d\alpha^+\right) - \left(\frac{\Gamma_0 - 2}{4}G(\alpha^-) + \frac{\Gamma_0}{8}\overline{F(\alpha^+)}\right)d\alpha^-\Big\} + O(\varepsilon^2) = 0, \end{aligned} \tag{131}$$

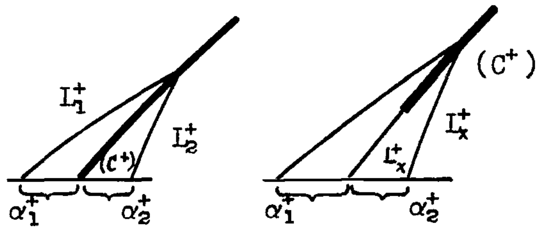
which demonstrates that $\frac{d\alpha^+}{d\alpha^-} = O(\epsilon)$ and thus permits to simplify the /V,60
writing of eq.(131), to arrive finally at

$$\frac{d\alpha^+}{d\alpha^-} = \frac{\varepsilon\Gamma_0}{4}\left\{\frac{\overline{F(\alpha^+)}}{2} - F(\alpha^+)\right\} \tag{132}$$

with, consequently,

$$\frac{d\,\overline{\alpha^+}}{d\alpha^-} = 0\,, \tag{133}$$

such that $\overline{\alpha^+}$ always remains constant along the shock. The base of the shock, or the foot of the characteristic which it prolongs, on $t = 0$, cut the segment (α_1^+, α_2^+) into two equal parts, to within $O(\epsilon^2)$.



Thus, the shock is defined parametrically by α^- while $\overline{\alpha^+}$ is used for individualizing this shock. We then have

$$\left\{ \begin{aligned} & \alpha_1^+ + \alpha_2^+ = \overline{\alpha^+}, \\ & \frac{\alpha_1^+ - \alpha_2^+}{F(\alpha_1^+) - F(\alpha_2^+)} = -\varepsilon \frac{\Gamma_0}{4}\left(\alpha^- - \frac{\overline{\alpha^+}}{2}\right), \end{aligned} \right. \tag{134}$$

so that the coordinates ξ, η of a point of the shock will be

$$\left\{ \begin{aligned} & \xi^+ = \frac{1}{2}\overline{\alpha^+} + \varepsilon\left\{\frac{\Gamma_0}{4}\left(\alpha^- - \frac{\overline{\alpha^+}}{2}\right)\frac{\overline{F(\alpha^+)}}{2} + \frac{\Gamma_0 - 2}{4}\int_{\alpha^-}^{\overline{\alpha^+}/2} G(z)\,dz\right\} + O(\varepsilon^2) \\ & \eta^+ = \alpha^- + \varepsilon\left\{\frac{\Gamma_0}{4}\left(\alpha^- - \frac{\overline{\alpha^+}}{2}\right)G(\alpha^-) \bullet \frac{\Gamma_0 - 2}{4}\int_{\alpha^-}^{\overline{\alpha^+}/2} F(z)\,dz\right\} + O(\varepsilon^2) . \end{aligned} \right. \tag{135}$$

Since $\alpha^- - \frac{\overline{\alpha^+}}{2}$ must be positive, the relation (134b) demonstrates that a /V,61
shock (C^+) could only correspond to pairs (α_1^+, α_2^+) located on an interval where the slope of the curve $F(\alpha^+)$ is negative. If, in such an interval, $F(\alpha^+)$ is continuous, the shock will be characterized by the position $\frac{1}{2}\overline{\alpha^+}$ where the slope reaches its maximum absolute value. Thus, if $F(\alpha^+)$ is twice continuously differentiable in a given interval, the quantities $\overline{\alpha^+}$ of this interval will be defined by

$$F''\left(\frac{\overline{\alpha^+}}{2}\right) = 0 \qquad F'\left(\frac{\overline{\alpha^+}}{2}\right) < 0 , \tag{136}$$

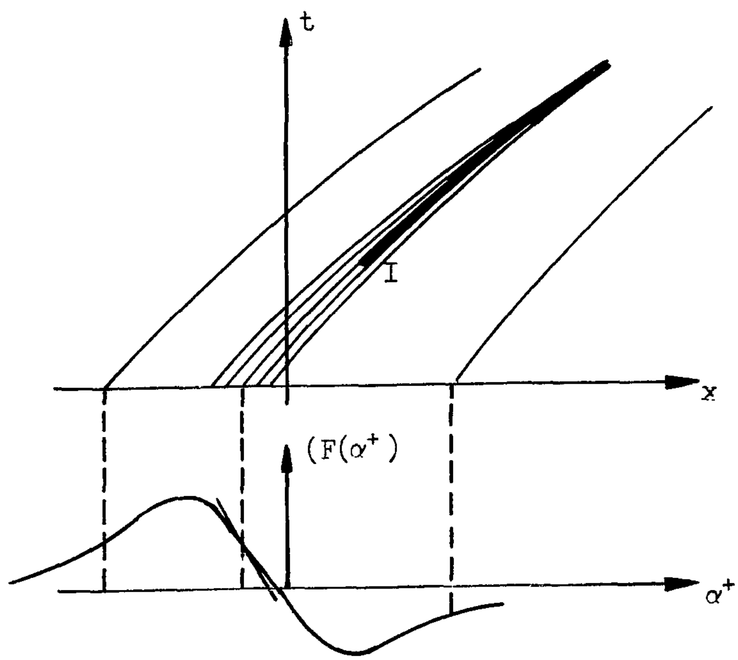
while the initial point of the shock will be defined by

$$
\begin{cases}
\xi_I^+ = \frac{1}{2}\hat{\alpha}^+ - F\left(\frac{\hat{\alpha}^+}{2}\right)\frac{1}{F'\left(\frac{\hat{\alpha}^+}{2}\right)} + O(\varepsilon) \\
\eta_I^+ = \frac{1}{2}\hat{\alpha}^+ - \frac{4}{\varepsilon\Gamma_0}\,\frac{1}{F'\left(\frac{\hat{\alpha}^+}{2}\right)}\left\{1 + O(\varepsilon)\right\},
\end{cases}
\tag{137}
$$

such that the initial instant of appearance of the shock will be defined by

$$
c_0 t_I = -\frac{2}{\varepsilon\Gamma_0 F'\left(\frac{\hat{\alpha}^+}{2}\right)} \tag{138}
$$

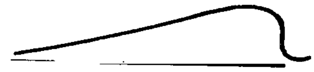
Thus, the shock appears earlier, the steeper the profile $F(\alpha^+)$. If the profile is infinitely steep, i.e., if it is discontinuous or if it is not differentiable,



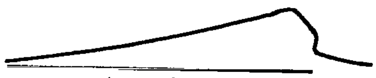
the shock will appear at the initial instant. The mechanism that leads to the formation of a shock C^+ is easy to understand. According to eq.(108), P^+ is maintained along a characteristic L^+; however, when $F'(\alpha^+) > 0$, the characteristics will converge toward each other in time, which leads to an increase of the profile of P^+ as a function of x at each instant; this becomes more pronounced with time and the shock will then appear already as soon as a vertical tangent appears in the profile. Conversely, when $F'(\alpha^+) > 0$, the characteristics L^+ will diverge and the profile of P^+ will spread. The accompanying diagram illustrates this particular mechanism. /V,62

t = 0

$t = t_1 < t_I$



$t = t_I$



$t > t_I$

What we stated above with respect to the shock (C^+) can be repeated for the shock (C^-), yielding

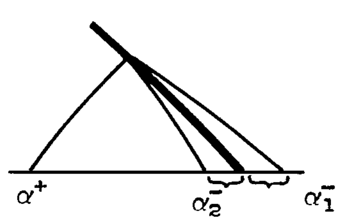
$$\left\{\begin{array}{l} \alpha_1^- + \alpha_2^- = \hat{\alpha}_- = \text{Const} \\ \dfrac{\alpha_1^- - \alpha_2^-}{G(\alpha_1^-) - G(\alpha_2^-)} = -\varepsilon \dfrac{\Gamma_0}{4}\left(\alpha^+ - \dfrac{\hat{\alpha}_-}{2}\right), \end{array}\right. \tag{139}$$

where the coordinates ξ, η are

/V,63

$$\left\{\begin{array}{l} \xi^- = \alpha^+ + \varepsilon\left\{\dfrac{\Gamma_0}{4}\left(\alpha^+ - \dfrac{\hat{\alpha}^-}{2}\right) F(\alpha^+) + \dfrac{\Gamma_0 - 2}{4}\displaystyle\int_{\frac{\hat{\alpha}^-}{2}}^{\alpha^+} G(\tau)\, d\tau\right\} + O(\varepsilon^2) \\ \eta^- = \dfrac{1}{2}\hat{\alpha}^- + \varepsilon\left\{\dfrac{\Gamma_0}{4}\left(\alpha^+ - \dfrac{\hat{\alpha}^-}{2}\right) G(\hat{\alpha}^-) + \dfrac{\Gamma_0 - 2}{4}\displaystyle\int_{\alpha^+}^{\frac{\hat{\alpha}^-}{2}} F(\tau)\, d\tau\right\} + O(\varepsilon^2). \end{array}\right. \tag{140}$$

If the function G comprises a part with a positive slope, a shock may be able to form. In addition, t = 0 is formed if G is discontinuous. If G is twice



continuously differentiable, we have

$$G''(\overline{\alpha^-}/2) = 0 , \quad G'(\overline{\alpha^-}/2) > 0 \tag{141}$$

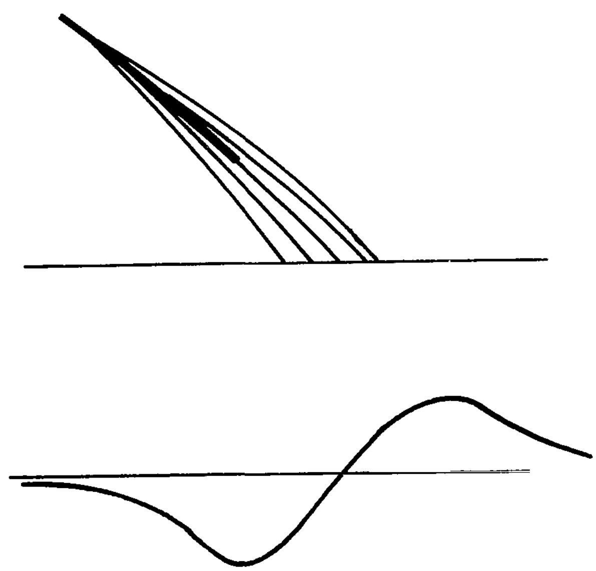
for determining $\overline{\alpha^-}$. The initial point of the shock is defined by

$$\begin{cases} \xi_I = \frac{1}{2}\overline{\alpha^-} - \frac{4}{\varepsilon\Gamma_0}\frac{1}{G'(\overline{\alpha^-}/2)}\{1+o(\varepsilon)\} \\ \eta_I = \frac{1}{2}\overline{\alpha^-} - G(\overline{\alpha^-}/2)\frac{1}{G'(\overline{\alpha^-}/2)} + o(\varepsilon) \end{cases} \tag{142}$$

while the initial instant is

$$c_0 t_I = \frac{2}{\varepsilon\Gamma_0 G'(\overline{\alpha^-}/2)}. \tag{143}$$

Thus, the known quantity of the functions $F(\alpha^+)$ and $G(\alpha^-)$ makes it possible, in first approximation and over a finite time interval, to place the shocks of



two families. Thus, without ambiguity, we can define to within $O(\varepsilon^2)$ the functions $\Delta^+(\alpha^+, \alpha^-)$ and $\Delta^-(\alpha^-, \alpha^+)$ which enter eq.(120). This yields corrections $O(\varepsilon^3)$ which must be included into ξ_2, η_2 such that the formal integration /V,64

of eq.(117) will result. To perform this calculation, we must define the properties of differentiability of F and G, in view of the fact that F' and G' are contained in $\frac{\partial\ \eta_1 - \xi_1}{\partial\ \alpha^{\pm}}$ which enter in eq.(117). If F and G are continuously differentiable in steps, we can still write eq.(117) but then the discontinuities of F and G in the integration must be taken into consideration. For simplification, let us assume that F and G are continuously differentiable, thus yielding

$$
\left\{
\begin{aligned}
\xi = {} & \alpha^+ + \varepsilon \left\{ \frac{\Gamma_0}{4}(\alpha^- - \alpha^+)\, F(\alpha^+) + \frac{\Gamma_0 - 2}{4} \int_{\alpha^+}^{\alpha^-} G(z)\, dz \right\} + \\
& + \varepsilon^2 \Bigg\{ - \frac{\Gamma_0}{4} F(\alpha^+) \sum [\underline{\alpha}^-]_1^- - \frac{\Gamma_0 - 2}{4} \sum [\underline{\alpha}^-\, G(\underline{\alpha}^-)]_1^- + \\
& + (\alpha^+ - \alpha^-) \left\{ \frac{\Gamma_0^2}{16} F(\alpha^+)\big(F(\alpha^+) + G(\alpha^-)\big) + \frac{\Gamma_0(\Gamma_0 - 2)}{32} \big(G(\alpha^-)\big)^2 \right\} + \\
& + \frac{\Gamma_0 - 2}{4} \int_{\alpha^-}^{\alpha^+} \left\{ \frac{\Gamma_0}{4} F(\alpha^+)\big(F(z) + 2G(z)\big) + \frac{3\Gamma_0 - 4}{8} \big(G(z)\big)^2 + \frac{\Gamma_0 - 2}{4} F(z) G(z) \right\} dz \Bigg\} \\
& + O(\varepsilon^3) \\
\eta = {} & \alpha^- + \varepsilon \left\{ \frac{\Gamma_0}{4}(\alpha^+ - \alpha^-)\, G(\alpha^-) + \frac{\Gamma_0 - 2}{4} \int_{\alpha^-}^{\alpha^+} F(t)\, dt \right\} + \\
& + \varepsilon^2 \Bigg\{ + \frac{\Gamma_0}{4} G(\alpha^-) \sum [\underline{\alpha}^+]_1^+ + \frac{\Gamma_0 - 2}{4} \sum [\underline{\alpha}^+\, F(\underline{\alpha}^+)]_1^+ + \\
& + (\alpha^- - \alpha^+) \left\{ \frac{\Gamma_0^2}{16} G(\alpha^-)\big(G(\alpha^-) + F(\alpha^+)\big) + \frac{\Gamma_0(\Gamma_0 - 2)}{32} \big(F(\alpha^+)\big)^2 \right\} + \\
& + \frac{\Gamma_0 - 2}{4} \int_{\alpha^+}^{\alpha^-} \left\{ \frac{\Gamma_0}{4} G(\alpha^-)\big(G(z) + 2F(z)\big) + \frac{3\Gamma_0 - 4}{8} \big(F(z)\big)^2 + \frac{\Gamma_0 - 2}{4} F(t) G(t) \right\} dz \Bigg\} + \\
& + O(\varepsilon^3) .
\end{aligned}
\right.
\tag{144}
$$

In these formulas, $\epsilon\Sigma\ [\alpha^-]_1$ denotes the sum of the discontinuities of α^- on traversing the shocks (C^-) such that the corresponding pairs $(\underline{\alpha}_1^-, \underline{\alpha}_2^-)$ are comprised in the interval (α^+, α^-) $(\alpha^+ \leq \underline{\alpha}_1^- \leq \underline{\alpha}_2^- \leq \underline{\alpha}_3^-)$; here, $\epsilon\Sigma\ [\underline{\alpha}^- G(\underline{\alpha}^-)]_1^-$ denotes the sum of the corresponding discontinuities of $\underline{\alpha}^- G(\underline{\alpha}^-)$ where the underlining indicates that a running variable is involved. The sense is such that $\epsilon[\underline{\alpha}^-]_1^- > 0$. A similar definition exists for $\epsilon\Sigma[\underline{\alpha}^+]_1^+$ and $\epsilon\Sigma[\alpha^+ F(\underline{\alpha}^+)]_1^+$ with, again, $[\underline{\alpha}^+]_1^+ > 0$. The subscript 1 in $[\ ]_1^{\pm}$ indicates that a discontinuity, evaluated to within $O(\epsilon^2)$, is involved here, meaning the discontinuity that had already been calculated above. For this, let us imagine that /V.65

$$[\underline{\alpha}^+]^+ = \varepsilon\, [\underline{\alpha}^+]^+_1 + \varepsilon^2 [\underline{\alpha}^+]^+_2 + \cdots \tag{145}$$

However, we do not know as yet how to evaluate $[\underline{\alpha}^+]^+_2$. The important point is the following: The localization of the shocks, at a certain order, makes it possible to calculate ξ and η at the following order in the hierarchy of approximations which, in turn, makes it possible to progress from one unity in the hierarchy of approximations with respect to the localization of shocks. To localize the shocks in second approximation, we write the continuity of ξ and that of η, as well as the formula analogous to eq.(130). Starting from this stage, the formulas become highly complicated but it seems that the procedure could be continued if F(z) and G(z) were continuously differentiable in steps up to an order corresponding to the desired approximation order. Naturally, starting with $O(\epsilon^3)$, eq.(108) must be improved which complicates the writing to such a degree that it is not even desirable to continue in this direction. It should be mentioned that, on progressing in the hierarchy of approximation, new shocks can be introduced whose intensity is of a higher order in ϵ than that of the shocks obtained in the preceding approximations, which greatly complicates the investigation. /V,66

5.2.3 Asymptotic Behavior

The solution obtained in the preceding Section is valid, in principle, only as long as

$$\varepsilon |\xi_1| \ll \xi_0 \qquad \text{etc.} \ldots \tag{146}$$

i.e., as long as

$$\frac{\varepsilon\, |\alpha^- - \alpha^+|\, |F(\alpha^+)|}{|\alpha^+|} \ll 1 \quad , \quad \frac{\varepsilon\, |\alpha^- - \alpha^+|\, |G(\alpha^-)|}{|\alpha^-|} \ll 1 \;. \tag{147}$$

Thus, it must be expected that the solution will no longer be valid if, on a fixed characteristic L^+ or L^-, we progress up to instants such that $\epsilon c_0 t = O(\underline{P})$ where $\underline{P}$ is the length scale of the phenomenon, i.e., the width of the interval of the x axis on which F and G assume their significant values. To study the behavior of the solution in such a region and beyond it, we will start a new investigation. For this, we will rewrite eqs.(72) by using ξ and η as independent variables, as defined in eq.(110), which yields

$$\left\{\begin{aligned} &\frac{\partial p^+}{\partial \eta} + \frac{1}{2}\left(\frac{u}{c_0} + \frac{c - c_0}{c_0}\right)\left(\frac{\partial p^+}{\partial \eta} + \frac{\partial p^+}{\partial \xi}\right) + \frac{1}{2}\left(\tau g_s - c\frac{\partial p^+}{\partial s}\right)\left(\frac{\partial s}{\partial \xi} + \frac{\partial s}{\partial \eta}\right) = 0 \;, \\ &\frac{\partial p^-}{\partial \xi} + \frac{1}{2}\left(\frac{u}{c_0} - \frac{c - c_0}{c_0}\right)\left(\frac{\partial p^-}{\partial \eta} + \frac{\partial p^-}{\partial \xi}\right) - \frac{1}{2}\left(\tau g_s - c\frac{\partial p^-}{\partial s}\right)\left(\frac{\partial s}{\partial \xi} + \frac{\partial s}{\partial \eta}\right) = 0 \;, \\ &-\frac{\partial s}{\partial \xi} + \frac{\partial s}{\partial \eta} + \frac{u}{c_0}\left(\frac{\partial s}{\partial \eta} + \frac{\partial s}{\partial \xi}\right) = 0 \;. \end{aligned}\right. \tag{148}$$

We wish to obtain the behavior of P^+, P^-, and S as soon as one of the variables ξ or η increases indefinitely. In fact, if the two variables increase indefinitely at the same time, then $F(\alpha^+)$ and $G(\alpha^-)$ tend to zero in eq.(147), and the conditions (146) remain proved. Thus, two different investigations must be made, depending on whether η increases indefinitely while ξ remains bound or whether the opposite situation occurs. In other words, in accordance with the terminology introduced in Chapter I for studying cylindrical waves, we will have two proximal asymptotic behaviors to be investigated. For the first behavior, let us pose /V,67

$$\xi = \rho\,\bar{\xi} \qquad \eta = \frac{1}{\theta}\,\rho\,\bar{\eta} \tag{149}$$

by assuming that

$$\theta \to 0 \qquad \bar{\xi}\,,\quad \bar{\eta} \tag{150}$$

On substitution in eq.(148), we obtain

$$\left\{\begin{aligned}
&\frac{\partial P^+}{\partial \bar{\eta}} + \left(\frac{u}{c_0} + \frac{c - c_0}{c_0}\right)\frac{\partial P^+}{\partial \bar{\xi}} + \frac{1}{2}\left(\tau g_s - c\,\frac{\partial P^+}{\partial S}\right)\frac{\partial S}{\partial \bar{\xi}} = \\
&\qquad = -\frac{1}{2}\left(\frac{u}{c_0} + \frac{c - c_0}{c_0}\right)\frac{\partial P^+}{\partial \bar{\eta}} - \frac{\theta}{2}\left(\tau g_s - c\,\frac{\partial P^+}{\partial S}\right)\frac{\partial S}{\partial \bar{\xi}} \\
&\frac{\partial P^-}{\partial \bar{\xi}} - \frac{1}{2}\left(\tau g_s - c\,\frac{\partial P^-}{\partial S}\right)\frac{\partial S}{\partial \bar{\xi}} = -\frac{1}{2}\left(\frac{u}{c_0} - \frac{c - c_0}{c_0}\right)\left(\frac{\partial P^-}{\partial \bar{\xi}} + \theta\,\frac{\partial P^-}{\partial \bar{\eta}}\right) \\
&\qquad + \frac{1}{2}\,\theta\left(\tau g_s - c\,\frac{\partial P^-}{\partial S}\right)\frac{\partial S}{\partial \bar{\eta}} \\
&\frac{\partial S}{\partial \bar{\xi}} = \frac{u}{c_0}\,\frac{\partial S}{\partial \bar{\xi}} + \theta\left(1 + \frac{u}{c_0}\right)\frac{\partial S}{\partial \bar{\eta}}\,.
\end{aligned}\right. \tag{151}$$

Since $\frac{u}{c_0}$, $\frac{c - c_0}{c_0}$, $\frac{P^+}{c_0}$, $\frac{P^-}{c_0}$, $\frac{S - S_0}{S_0}$, $\mathfrak{D}$ are small, the second members of eq.(151) will be negligible compared to the corresponding first members. /V,68
The last equation in the system (151) yields $\frac{\partial S}{\partial \bar{\xi}} = 0$, i.e., $S(\bar{\xi}, \bar{\eta}) = S(\bar{\eta})$ such that, for large $\bar{\eta}$ along η = const, the quantity $S - S_0$ will have the same value as for large ξ, i.e.,

$$\frac{S - S_0}{S_0} \leq O(\varepsilon^3) \tag{152}$$

is valid even for large time values. This seems to indicate that $\left|\frac{S - S_0}{S_0}\right| \ll \ll \frac{|p^+|}{c_0}$, whereas the second equation of the system (151) shows that $\frac{|p^-|}{c_0} = = 0\left(\left|\frac{S - S_0}{S_0}\right|\right)$. Returning to the first equation of the system (151), we obtain

$$\left|\frac{u}{c_0} + \frac{c - c_0}{c_0}\right| = 0\,(\theta) \tag{153}$$

However, according to the above statement and according to eq.(81), we have

$$\frac{u}{c_0} + \frac{c - c_0}{c_0} = \frac{\Gamma_0}{2}\frac{p^+}{c_0} + \frac{\Lambda_0}{8}\left(\frac{p^+}{c_0}\right)^2 + 0\left\{\left(\frac{|p^+|}{c_0}\right)^3 + \frac{|p^-|}{c_0} + \frac{|S - S_0|}{S_0}\right\}, \tag{154}$$

such that we must pose

$$p^+ = c_0\,\theta\,\bar{V} \tag{155}$$

and assume that

$$\lim_{\theta \to 0} \bar{V} = \bar{V}_0 \qquad \text{finite,} \tag{156}$$

which yields, for $\bar{V}_0$, the following equation:

$$\frac{\partial \bar{V}_0}{\partial \bar{\eta}} + \frac{\Gamma_0}{4}\bar{V}_0\frac{\partial \bar{V}_0}{\partial \xi} = 0 \tag{157}$$

whose solution is readily obtained in the form of

$$\begin{cases} \bar{V}_0 = \bar{V}_0^{(0)}(\omega), \\ \xi = \omega + \frac{\Gamma_0}{4}\bar{V}_0^{(0)}(\omega)\,\bar{\eta}. \end{cases} \tag{158}$$

Let us now return to the solution of the preceding Section /V,69

$$\begin{cases} p^+ = \varepsilon c_0\, F(\alpha^+), \qquad p^- = \varepsilon c_0\, G(\alpha^-), \\ \xi = \alpha^+ + \varepsilon\left\{\frac{\Gamma_0}{4}(\alpha^- - \alpha^+)F(\alpha^+) + \frac{\Gamma_0 - 2}{4}\int_{\alpha^+}^{\alpha^-} G(t)\,dt\right\}, \end{cases} \tag{159}$$

$$\eta = \alpha^- + \varepsilon \left\{ \frac{p_0}{4}(\alpha^+ - \alpha^-) G(\alpha^-) + \frac{p_0 - 2}{4} \int_{\alpha^-}^{\alpha^+} F(t)\, dt \right\}$$

and let us define its behavior as $\alpha^- \to \infty$. We assume here that $\alpha^- G(\alpha^-) \to 0$, so that we have

$$\left\{ \begin{array}{ll} P^+ = \varepsilon c_0 F(\alpha^+) & |P^-| \leq c_0\, O\left(\frac{\varepsilon}{|\alpha^-|}\right) \\ \xi \cong \alpha^+ + \varepsilon \frac{p_0}{4}(\alpha^- - \alpha^+) F(\alpha^+) + O(\varepsilon\alpha^+) & \\ \eta \cong \alpha^- + O\left(\varepsilon\alpha^+ \int_{-\infty}^{\infty} F(z) \frac{dz}{\alpha^+}\right) & \end{array} \right. \tag{160}$$

If $O(\varepsilon)$ is neglected with respect to 1, it will be seen that eq.(160) contains eqs.(155) and (158). Thus, the solution (159) is actually valid even for large values of time, since it contains the principal term of the proximal solution. Conversely, the method of localization of shocks, which was used by us in the preceding Section, is no longer valid. Thus, it is also no longer legitimate to derive from eq.(128) that $\alpha_2^+ - \alpha_1^+ = O\{\varepsilon(\alpha_1^+ + \alpha_2^+)\}$ as soon as $\varepsilon\left(\alpha^- - \frac{\overline{\alpha^+}}{2}\right) = O\left(\frac{\overline{\alpha^+}}{2}\right)$ which is the case here. Naturally, the investigation of the shocks can be made on the basis of eq.(159) by considering that $(\alpha^- - \alpha^+) F(\alpha^+)$ may be of the same order as α^+; however, it is simpler to base the investigation on eq.(157), since eq.(159) contained terms that are negligible in the proximal behavior and since, when working with eq.(159), useless terms will interfere. In addition, from another viewpoint, the procedure consisting in looking for a proximal representation by means of eq.(157) is much more satisfactory. /V,70
In fact, if we set $\varepsilon \frac{\alpha^- - \alpha^+}{\alpha^+} = O(1)$, we have difficulty in conceiving why the expansion (159) should be limited to its two first terms and, a priori, it is quite impossible to understand why $\varepsilon^2\xi_2$ does not contain terms $O(1)$ under such conditions. In fact, this is actually not so, as indicated by eq.(144); however, this can be obtained only from an explicit calculation and it is by no means certain that this fortunate circumstance will be maintained on extending the series beyond $\varepsilon^2\xi_2$. If, in addition, $\varepsilon \frac{\alpha^- - \alpha^+}{\alpha^+} \gg 1$, we also have $\varepsilon\xi_1 \gg \xi_0$ so that the notion of an expansion by the small-parameter method loses its entire sense. Conversely, if we return to the operations that led to eq.(157), it is obvious that it had not been necessary to define the order of θ relative

to ϵ and that it had been sufficient that $\theta \ll 1$. This fact can be used profitably. For example, if we select $\theta = \sqrt{\epsilon}$, we will have $\dfrac{\epsilon|\alpha^- - \alpha^+|}{\alpha^+} = O(\sqrt{\epsilon})$, i.e., we will be in a domain in which $\epsilon|\xi_1| \ll \xi_0$, i.e., in the domain where eq.(159) can be again valid. From this, the connectivity law is derived: So that the proximal representation will extend the representation obtained in the preceding Section, it is suitable to select, from the solutions of eq.(157), the one solution that is identical with the solution derived from eq.(159), including the shock, when setting $\theta = \sqrt{\epsilon}$ in the former and setting $\alpha^- = \epsilon^{-1/2} - \alpha^-$ in the latter. The identification must be made between the principal terms /V,71 as $\epsilon \to 0$. Before attempting to go further in this direction, let us return to eqs.(3) and let us write these equations in the variables ξ, η; let us then change eq.(149) and disregard any term which is small with respect to an already retained term since it contains a factor being either θ, or $\dfrac{u}{c_0}$, or $\dfrac{p^\pm}{c_0}$, or $\dfrac{S - S_0}{S_0}$; we will then obtain

$$\left\{\begin{aligned} &\frac{\partial p^+}{\partial \eta} + \frac{1}{2\theta}\left(\frac{u}{c_0} + \frac{c-c_0}{c_0}\right)\frac{\partial p^+}{\partial \xi} + \frac{1}{2\theta}\frac{\tau_0}{c_0} g_{s_0}\frac{\partial s}{\partial \xi} - \frac{1}{4\theta}\left(\frac{4}{3}\mu_0+\mu_{v_0}\right)\frac{\tau_0}{c_0 \ell}\frac{\partial^2 (p^+ - p^-)}{\partial \xi^2} = 0, \\ &\frac{\partial p^-}{\partial \xi} + \frac{1}{2}\frac{\tau_0}{c_0} g_{s_0}\frac{\partial s}{\partial \xi} - \frac{1}{4}\left(\frac{4}{3}\mu_0+\mu_{v_0}\right)\frac{\tau_0}{c_0 \ell}\frac{\partial^2 (p^+ - p^-)}{\partial \xi^2} = 0, \\ &\frac{\partial s}{\partial \xi} + \frac{\tau_0 k_0}{T_0 c_0 \ell}\frac{\partial^2 T}{\partial \xi^2} = 0, \end{aligned}\right. \tag{161}$$

for which we must use

$$T = T_0 + \frac{1}{2}\tau_0 g_{s_0}\frac{p^+ + p^-}{c_0} + T_{s_0}(s - s_0). \tag{162}$$

Taking eq.(162) into consideration, the two last relations of the system (161) will integrate as follows:

$$\left\{\begin{aligned} &p^- + \frac{1}{2}\frac{\tau_0}{c_0} g_{s_0}(s-s_0) - \frac{1}{4}\left(\frac{4}{3}\mu_0+\mu_{v_0}\right)\frac{\tau_0}{c_0\ell}\frac{\partial (p^+ - p^-)}{\partial \xi} = \mathrm{Const}(\eta) \\ &s - s_0 + \frac{\tau_0 k_0}{T_0 c_0 \ell} T_{s_0}\frac{\partial (s-s_0)}{\partial \xi} + \frac{1}{2}\tau_0 g_{s_0}\frac{\tau_0 k_0}{T_0 c_0 \ell}\frac{\partial (p^+ + p^-)}{\partial \xi} = \mathrm{Const}(\eta) \end{aligned}\right. \tag{163}$$

and the constants in ξ must be taken as zero, which becomes obvious when evaluating the first terms for large ξ. Let us then pass to the dimensionless variables (4) and make use of eq.(13), yielding the following system [we denote by ϵ_v the expression denoted by ϵ in eq.(13), so as to obtain a differentiation from the parameter ϵ used in eq.(108) and later]:

$$\left\{\begin{aligned}
&\frac{\partial U}{\partial \eta} + \frac{1}{4\theta}(U+V)\frac{\partial U}{\partial \xi} + \frac{1}{2\chi_0\theta}\frac{\partial \Sigma}{\partial \xi} - \frac{\varepsilon_V}{4\theta}\frac{\partial^2 U - V}{\partial \xi^2} = 0,\\
&V + \frac{\varepsilon_V}{4}\frac{\partial V}{\partial \xi} + \frac{1}{2\chi_0}\Sigma - \frac{\varepsilon_V}{4}\frac{\partial U}{\partial \xi} = 0,\\
&\Sigma + \frac{\varepsilon_V K_0}{\lambda_0 - 1}\frac{\partial \Sigma}{\partial \xi} + \frac{\varepsilon K_0}{2}\frac{\partial U+V}{\partial \xi} = 0,
\end{aligned}\right. \tag{164}$$

Note: replace $U+V$ by $\rho_0 U + (\rho_0 - 2)V$ (error)

where the situation we are investigating here is characterized by

$$|U| \ll 1, \quad |V| \ll 1, \quad |\Sigma| \ll 1, \quad \theta \ll 1, \quad \varepsilon \ll 1. \tag{165}$$

Let us consider U as known and the two last equations in the system (164) as an ordinary differential system in V and Σ. Let us reduce this system to the canonical form by searching for α and β in such a manner that

$$\left\{\begin{aligned}
&\frac{\alpha}{4} + \beta\frac{K_0}{2} = r\alpha,\\
&\beta\frac{K_0}{\lambda_0 - 1} = r\left(\frac{\alpha}{2\chi_0} + \beta\right)
\end{aligned}\right. \tag{165a}$$

It will be found that r must verify the equation

$$r^2 + \left(\frac{K_0}{4\chi_0} - \frac{1}{4} - \frac{K_0}{\lambda_0 - 1}\right)r + \frac{K_0}{4(\lambda_0 - 1)} = 0, \tag{166}$$

whose two roots r_1 and r_2, which generally are real, have positive real parts under the condition that

$$\frac{1}{4} + \frac{4\chi_0 - \lambda_0 + 1}{4\chi_0(\lambda_0 - 1)}K_0 > 0, \tag{167}$$

to which roots α_1, β_1 and α_2, β_2 correspond. This yields

$$\alpha_i V + \left(\frac{\alpha_i}{2\chi_0} + \beta_i\right)\Sigma + \varepsilon_V r_i \frac{\partial}{\partial \xi}\left\{\alpha_i V + \left(\frac{\alpha_i}{2\chi_0} + \beta_i\right)\Sigma\right\} = \varepsilon_V\left(\frac{\alpha_i}{4} - \frac{\beta_i K_0}{2}\right)\frac{\partial U}{\partial \xi} \tag{168}$$

and, taking the fact that V and Σ vanish into consideration, we obtain /V,73

$$\alpha_i V + \left(\frac{\alpha_i}{2X_0} + \beta_i\right)\Sigma = \frac{\alpha_i - 2\beta_i K_0}{4\, n_i}\int_{-\infty}^{\bar{\xi}} e^{-\frac{1}{\varepsilon_v n_i}(\bar{\xi}-\bar{\xi}_1)} \frac{\partial U(\bar{\xi}_1, \bar{\eta})}{\partial \bar{\xi}}\, d\bar{\xi}_1 \qquad (169)$$

Since ε_v is very small, it follows that

$$\alpha_i V + \left(\frac{\alpha_i}{2X_0} + \beta_i\right)\Sigma = \varepsilon_v \frac{(\alpha_i - 2\beta_i K_0)}{4} \frac{\partial U(\bar{\xi}, \bar{\eta})}{\partial \bar{\xi}} - \frac{\varepsilon_v(\alpha_i - 2\beta_i K_0)}{4}\int_{-\infty}^{\bar{\xi}} e^{-\frac{1}{\varepsilon n_i}(\bar{\xi}-\bar{\xi}_1)} \frac{\partial^2 U(\bar{\xi}_1, \bar{\eta})}{\partial \bar{\xi}^2}\, d\bar{\xi}_1, \qquad (170)$$

so that

$$\{|V|, |\Sigma|\} \sim O\left(\varepsilon_v \left|\frac{\partial U}{\partial \bar{\xi}}\right|\right). \qquad (171)$$

We will temporarily admit that

$$\varepsilon_v \left|\frac{\partial U}{\partial \bar{\xi}}\right| \ll U, \qquad (172)$$

in view of the working hypothesis which we are attempting to prove a posteriori. Under these conditions, the last equation of the system (164) yields

$$\frac{1}{2X_0}\frac{\partial \Sigma}{\partial \xi} = -\frac{\varepsilon_v K_0}{4X_0}\frac{\partial^2 U}{\partial \xi^2}, \qquad (173)$$

such that the first equation of the system (164) becomes

$$\frac{\partial U}{\partial \bar{\eta}} + \Gamma_0 \frac{U}{4\theta}\frac{\partial U}{\partial \bar{\xi}} - \frac{\varepsilon_v}{4\theta}\left(1 + \frac{K_0}{X_0}\right)\frac{\partial^2 U}{\partial \bar{\xi}^2} = 0. \qquad (174)$$

Let us recall now that

$$\theta \ll 1, \quad U \ll 1, \qquad (175)$$

which indicates that the nonlinear term is comparable to the linear terms in eq.(174) only if θ and U are of the same order, which leads us to assume /V,74

$$\bar{U} = \theta \, \bar{U}_{0\nu} + \ldots \tag{175a}$$

and to keep ε_ν and θ infinitely small (independent), thus yielding

$$\frac{\partial \bar{U}_{0\nu}}{\partial \bar{\eta}} + \frac{\Gamma_0}{4} \bar{U}_{0\nu} \frac{\partial \bar{U}_{0\nu}}{\partial \bar{\xi}} - \frac{\varepsilon_\nu}{4\theta}\left(1 + \frac{K_0}{\chi_0}\right) \frac{\partial^2 \bar{U}_{0\nu}}{\partial \bar{\xi}^2} = 0 , \tag{176}$$

which must be compared to eq.(157). It is obvious that, taking the dissipation phenomena and the nonlinear convection phenomena into consideration, the behavior of a plane wave train (proximal asymptotic behavior) is governed by a Burgers equation.

Burgers' equation:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = \nu \frac{\partial^2 u}{\partial x^2} , \tag{177}$$

whose solution is obtained by writing

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x}\left(\frac{u^2}{2} - \nu \frac{\partial u}{\partial x}\right) = 0 , \tag{178}$$

i.e.,

$$u = \frac{\partial \phi}{\partial x} , \qquad \nu \frac{\partial u}{\partial x} - \frac{u^2}{2} = \frac{\partial \phi}{\partial t} , \tag{179}$$

from which, posing

$$\phi = -2\nu \operatorname{Log} \psi , \tag{180}$$

we obtain

$$\frac{\partial \psi}{\partial t} = \nu \frac{\partial^2 \psi}{\partial x^2} . \tag{181}$$

Equation (181) admits a solution of the form /V,75

$$\psi(t, x) = \frac{1}{\sqrt{4\pi \nu (t - t_0)}} \int_{-\infty}^{\infty} \psi(t_0, \xi) \exp\left\{ - \frac{(x - \xi)^2}{4\nu(t - t_0)} \right\} d\xi . \tag{182}$$

We have

$$\Psi(t_0, x) = \exp\left\{-\frac{1}{2\nu}\int_{-\infty}^{x} u(t_0, \xi)\, d\xi\right\}, \tag{183}$$

and, consequently,

$$u = -2\nu_e \frac{\partial \operatorname{Log}\Psi}{\partial x}. \tag{184}$$

Thus,

$$u(t,x) = \frac{\displaystyle\int_{-\infty}^{\infty} \frac{x-\xi}{t-t_0} \exp\left\{-\frac{1}{2\nu}\int_{-\infty}^{\xi} u(t_0, \eta)\, d\eta - \frac{(x-\xi)^2}{4\nu(t-t_0)}\right\} d\xi}{\displaystyle\int_{-\infty}^{\infty} \exp\left\{-\frac{1}{2\nu}\int_{-\infty}^{\xi} u(t_0, \eta)\, d\eta - \frac{(x-\xi)^2}{4\nu(t-t_0)}\right\} d\xi} \tag{185}$$

is a solution of Burgers' equation. This permits finding a solution of eq.(176):

$$\bar{U}_{0v}(\bar{\xi}, \bar{\eta}) = \frac{\displaystyle\int_{-\infty}^{\infty} \frac{4(\bar{\xi}-\bar{\xi}_1)}{\Gamma_0(\bar{\eta}-\bar{\eta}_0)} \exp\left\{-\frac{\Gamma_0\theta}{2\mathcal{E}_v}(1+K_\gamma/X_0) F(\bar{\xi}, \bar{\xi}_1, \bar{\eta}, \bar{\eta}_0)\right\} d\bar{\xi}_1}{\displaystyle\int_{-\infty}^{\infty} \exp\left\{-\frac{\Gamma_0\theta}{2\mathcal{E}_v}\left(1+\frac{K_\gamma}{X_0}\right) F(\bar{\xi}, \bar{\xi}_1, \bar{\eta}, \bar{\eta}_0)\right\} d\bar{\xi}_1}, \tag{186}$$

with

$$F(\bar{\xi}, \bar{\xi}_1, \bar{\eta}, \bar{\eta}_0) = \int_{-\infty}^{\bar{\xi}_1} \bar{U}_{0v}(\bar{\xi}_2, \bar{\eta}_0)\, d\bar{\xi}_2 + \frac{2(\bar{\xi}-\bar{\xi}_1)^2}{\Gamma_0(\bar{\eta}-\bar{\eta}_0)}. \tag{187}$$

To have eqs.(186) and (187) solve the problem of the proximal behavior, it is necessary to know the value of $\bar{U}_{0v}(\bar{\xi}, \bar{\eta}_0)$. For this, we will make use of the connectivity law, stipulated on p.307. Primarily, we state that it is legitimate to neglect the viscosity in the domain in which the connection is made. In fact, in this domain, we have $\theta = \sqrt{\varepsilon}$ and $\ell = L$ where L is the length scale for the initial distribution (73), such that /V,76

$$\frac{\mathcal{E}_V}{\theta} \cong O\left(\frac{\tau_0 \mu_0}{L c_0} \sqrt{\frac{c_0}{u_{in}}}\right) = \frac{1}{\sqrt{Re_{c_0} \, Re_{u_{in}}}} , \qquad (188)$$

where Re_{c_0} and $Re_{u_{in}}$ are the Reynolds numbers

$$Re_{c_0} = \frac{\rho_0 c_0 L}{\mu_0} , \qquad Re_{u_{in}} = \frac{\rho_0 u_{in} L}{\mu_0} , \qquad (189)$$

(here, u_{in} is a characteristic value for the velocity u at the initial instant). Since the two Reynolds numbers (189) are very large, we naturally have

$$\left(\frac{\mathcal{E}_V}{\theta}\right)_{\text{connectivity}} \ll 1 . \qquad (190)$$

Therefore, it was legitimate to make use of eq.(157) in the connectivity region. That we took the dissipation effects into consideration is due to the fact that we were not a priori certain that it is always legitimate to neglect these effects in the proximal domain if θ is taken at increasingly smaller values. We will demonstrate that this is exactly so.

Let us first mention that we can arbitrarily select ℓ and θ in eq.(149) provided that $\theta \ll 1$ and that

$$\mathcal{E}_V = \left(\frac{4}{3} \mu_0 + \mu_{v_0}\right) \tau_0 \ell^{-1} c_0^{-1} , \qquad (191)$$

as well as that we have eq.(175) and that eq.(172) is proved. Then, eq.(176) continues to govern the phenomenon. This means that the validity of eq.(186) is not limited to a finite $\bar{\eta}$. For each value of $\bar{\xi}$ and $\bar{\eta}$, the quantity F is a function of $\bar{\xi}_1$ which is continuous, is bounded by a finite $\bar{\xi}_1$, and increases indefinitely as $|\bar{\xi}_1| \to \infty$, provided (which we assume here) that /V,77

$$\left| \int_{-\infty}^{\xi} \bar{U}_{0v}(\bar{\xi}, \bar{\eta}_0) \, d\bar{\xi} \right| < \infty , \qquad (188)$$

such that F will reach its lower bound F^* for a (not necessarily unique) value of $\bar{\xi}_1$, i.e., $\bar{\xi}^*$. Let us assume here that $U_{0v_0} = U_{0v}(\bar{\xi}, \bar{\eta}_0)$ is twice continuously differentiable in the vicinity of $\bar{\xi}^*$, thus yielding

$$\begin{cases} \bar{U}^{*}_{0V_0} - \dfrac{4(\bar{\xi}-\bar{\xi}^{*})}{P_0(\bar{\eta}-\bar{\eta}_0)} = 0, \\ F - F^{*} = \dfrac{1}{2}\left\{\bar{U}'^{*}_{0V_0} + \dfrac{4}{P_0(\bar{\eta}-\bar{\eta}_0)}\right\}(\bar{\xi}_1-\bar{\xi}^{*})^2 + \dfrac{\bar{U}''^{*}_{0V_0}}{3}(\bar{\xi}_1-\bar{\xi}^{*})^3 + \cdots, \end{cases} \tag{189}$$

so that eq.(186) becomes

$$\bar{U}_{0V}(\bar{\xi},\bar{\eta}) = \bar{U}_{0V}(\bar{\xi}^{*},\bar{\eta}_0) + \frac{\int_{-\infty}^{\infty}\frac{4}{P_0}\frac{\bar{\xi}^{*}-\bar{\xi}_1}{\bar{\eta}-\bar{\eta}_0}\exp\left\{-\frac{P_0\theta}{2\varepsilon_V}\left(1+\frac{K_0}{\chi_0}\right)(F-F^{*})\right\}d\bar{\xi}_1}{\int_{-\infty}^{\infty}\exp\left\{-\frac{P_0\theta}{2\varepsilon_V}\left(1+\frac{K_0}{\chi_0}\right)(F-F^{*})\right\}d\bar{\xi}_1}. \tag{190}$$

For asymptotically evaluating the second integral, we write

$$\exp\{\ \} = \left\{1 - \frac{P_0\theta}{6\varepsilon_V}\left(1+\frac{K_0}{\chi_0}\right)\bar{U}''^{*}_{0V_0}(\bar{\xi}_1-\bar{\xi}^{*})^3 + \cdots\right\}\cdot$$
$$\cdot\ \exp\left\{-\frac{P_0\theta}{4\varepsilon_V}\left(1+\frac{K_0}{\chi_0}\right)\left(\bar{U}'^{*}_{0V_0}+\frac{4}{(\bar{\eta}-\bar{\eta}_0)P_0}\right)(\bar{\xi}_1-\bar{\xi}^{*})^2\right\} \tag{191}$$

whence

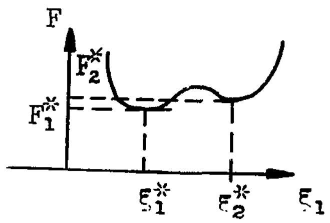
$$\begin{cases} \bar{U}_{0V}(\bar{\xi},\bar{\eta}) = \bar{U}^{*}_{0V_0} + \dfrac{8}{P_0^2}\dfrac{\varepsilon_V}{\theta}\dfrac{\bar{U}''^{*}_{0V_0}}{\left(1+\frac{K_0}{\chi_0}\right)\left\{\bar{U}'^{*}_{0V_0}+\frac{4}{P_0(\bar{\eta}-\bar{\eta}_0)}\right\}}\dfrac{1}{\bar{\eta}\,\bar{\eta}_0} + \cdots. \\ \bar{\xi} = \bar{\xi}^{*} + \dfrac{P_0}{4}(\bar{\eta}-\bar{\eta}_0)\bar{U}^{*}_{0V_0}, \end{cases} \tag{192}$$

which permits formulating the following conditions: /V,78

$$\frac{\varepsilon_V\,\bar{U}''^{*}_{0V_0}}{\theta\ \bar{U}^{*}_{0V_0}\left(1+\frac{P_0}{4}(\bar{\eta}-\bar{\eta}_0)\bar{U}'^{*}_{0V_0}\right)} \ll 1 \tag{193}$$

so that the dissipation effects will become negligible outside of the shocks. It then remains to check whether the condition (172) is verified in the internal

structure of the shocks where $\left|\frac{\partial \bar{U}_{0_v}}{\partial \bar{\xi}}\right| \gg |\bar{U}_{0_v}|$. Thus, it is necessary to study the internal structure of the shocks, i.e., the regions where eq.(189) ceases being valid since F presents two neighboring relative minima F_1^* and F_2^*. For an



asymptotic evaluation of $\bar{U}_{0_v}$ in this case, we make use of eq.(189) in the vicinity of each minimum, inducing us to pose (i = 1, 2)

$$A_i = \int_{-\infty}^{\infty} \exp\left\{-\frac{r_0\theta}{4\varepsilon_v}\left(1+\frac{K_0}{\chi_0}\right)\left(\bar{U}'^{*}_{0V_0 i} + \frac{4}{(\bar{\eta}-\bar{\eta}_0)r_0}\right)(\bar{\xi}_1-\bar{\xi}_i^*)^2\right\} d\bar{\xi}_1 \tag{194}$$

from which we find

$$\begin{cases} U_{0_V}(\bar{\xi},\bar{\eta}) \simeq \dfrac{4}{r_0}\,\dfrac{B_1(\bar{\xi}_1^*-\bar{\xi}) + B_2(\bar{\xi}_2^*-\bar{\xi})}{(B_1+B_2)(\bar{\eta}-\bar{\eta}_0)}, \\ B = A\exp\left\{-\dfrac{r_0\theta}{2\varepsilon_V\left(1+\frac{K_0}{\chi_0}\right)}F^*\right\}, \end{cases} \tag{195}$$

with

$$A = \frac{\text{Const}}{\sqrt{\frac{4}{r_0} + (\bar{\eta}-\bar{\eta}_0)\,\bar{U}'^{*}_{0V_0}}}\sqrt{\bar{\eta}-\bar{\eta}_0}\,. \tag{196}$$

Using these formulas, we can describe the internal structure of the shocks. /V,79
If ϵ_v is rigorously equal to zero, we will also rigorously have

$$\begin{cases} \bar{U}_{0_V}(\bar{\xi},\bar{\eta}) = U^*_{0V_0}, \end{cases} \tag{197}$$

$$\bar{\xi} = \bar{\xi}^{*} + \frac{\Gamma_0}{4}(\bar{\eta} - \bar{\eta}_0)\, U_{0v_0}^{*},$$

where $\bar{U}_{0v}$ is discontinuous as soon as $\bar{\xi}$, at fixed $\bar{\eta}$, traverses a value for which F presents two equal relative minima at its lower bound. Let $\bar{\xi}_c(\bar{\eta})$ be the value of $\bar{\xi}$ for which this fact takes place and let $\bar{\xi}_i^{**}$ be the corresponding values with F^{**} for the common value of the minima. Let us assume that $\bar{U}_{0v_0}$ is regular in the vicinity of $\bar{\xi}_i^{**}$. If $\bar{\xi}$ is close to ξ_c, then F presents two relative minima F_1^* and F_2^* in $\bar{\xi}_1^*$ and $\bar{\xi}_2^*$ close to $\bar{\xi}_1^{**}$ and $\bar{\xi}_2^{**}$, which are defined by the first equation of the system (189), i.e.,

$$\left\{1 + \frac{\Gamma_0}{4}(\bar{\eta} - \bar{\eta}_0)\, \bar{U}_{0v_{0i}}^{\prime **}\right\}\left(\bar{\xi}^{*} - \bar{\xi}_i^{**}\right) = \bar{\xi} - \bar{\xi}_c. \tag{198}$$

The values of the minima are given by

$$F^{*} = F^{**} + (\bar{\xi} - \bar{\xi}_c)\, U_{0v_{0i}}^{**} \tag{199}$$

as it results from eq.(187), because of

$$\left(\frac{\partial F}{\partial \bar{\xi}_1}\right)^{**} = 0, \quad \left(\frac{\partial F}{\partial \bar{\xi}}\right)^{**} = U_{0v_{0i}}^{**}. \tag{200}$$

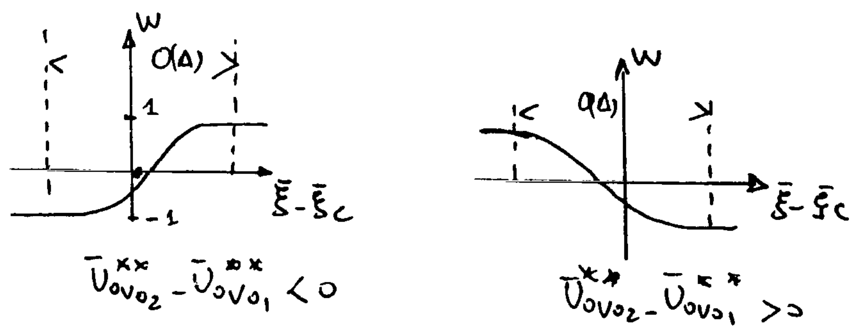
Application of eq.(195) immediately yields the asymptotic value of $\bar{U}_{0v}$ in the vicinity of ξ_c, i.e., in the internal structure of the shock: /V,80

$$\left\{\begin{aligned} U_{0v}(\bar{\xi}, \bar{\eta}) &= \frac{\bar{U}_{0v_{01}}^{*} + \bar{U}_{0v_{02}}^{*}}{2} + W\, \frac{\bar{U}_{0v_{02}}^{*} - \bar{U}_{0v_{01}}^{*}}{2} \\ W &= \frac{1 - \sqrt{\dfrac{1 + \frac{\Gamma_0}{4}(\bar{\eta} - \bar{\eta}_0)\, \bar{U}_{0v_{02}}^{\prime *}}{1 + \frac{\Gamma_0}{4}(\bar{\eta} - \bar{\eta}_0)\, U_{0v_{01}}^{\prime *}}}\, \exp\left\{-\frac{\Gamma_0\theta}{2\varepsilon_V}\left(1 + \frac{K_0}{\chi_0}\right)\left(\bar{U}_{0v_{02}}^{**} - \bar{U}_{0v_{01}}^{**}\right)(\bar{\xi} - \bar{\xi}_c)\right\}}{1 + \sqrt{\dfrac{1 + \frac{\Gamma_0}{4}(\bar{\eta} - \bar{\eta}_0)\, U_{0v_{02}}^{\prime *}}{1 + \frac{\Gamma_0}{4}\, \bar{U}_{0v_{01}}^{\prime *}(\bar{\eta} - \bar{\eta}_0)}}\, \exp\left\{\frac{\Gamma_0\theta}{2\varepsilon_V}\left(1 + \frac{K_0}{\chi_0}\right)\left(\bar{U}_{0v_{02}}^{**} - \bar{U}_{0v_{01}}^{**}\right)(\bar{\xi} - \bar{\xi}_c)\right\}} \end{aligned}\right. \tag{201}$$

It is obvious that W varies between -1 and +1 when $\bar{\xi}$ describes a small interval spread around ξ_c and having a width of O(Δ) with

$$\Delta = \frac{\varepsilon_v}{\theta} \frac{2\chi_0}{\rho_0(K_0+\chi_0)} \frac{1}{|\bar{U}^{**}_{0v_{02}} - \bar{U}^{**}_{0v_{01}}|} \tag{202}$$

In the interval in question, U_{0_v} passes from $U^{**}_{0_{v01}}$ to $U^{**}_{0_{v02}}$. Let us mention - in passing - that, since the interval Δ is very small, it is legitimate to



make use of eq.(199), i.e., not to differentiate between $U^{**}_{0_{v01}}$ and $U^{*}_{0_{v01}}$. /V,81

Equation (201) satisfactorily describes an internal structure in that it causes, in a very narrow zone, a rapid transition between the two faces of the discontinuity corresponding to $\varepsilon_v = 0$. Recalling that

$$\bar{\xi} = \frac{x - c_0 t}{\ell}, \qquad \bar{\eta} = \theta\, \frac{x + c_0 t}{\ell}, \tag{203}$$

it is obvious that, at fixed $\bar{\eta}$, an increasing $\bar{\xi}$ corresponds approximately to an increasing x at fixed t, i.e., to the direction of propagation of the shock. The sketch in the accompanying diagram shows that, on traversing the shock in the same direction as the fluid, i.e., from $\bar{\xi} - \bar{\xi}_c > 0$ toward $\bar{\xi} - \bar{\xi}_c < 0$ (since the fluid is practically at rest while the shock advances at a velocity very close to c_0), the quantity $\underline{P}^+$ increases. Since P^- and Σ vary rather little, this means that the shock is a compression discontinuity, as should be the case. Let us now return to the condition (172); eq.(201) shows that, in the internal structure of the shock, we have

$$\varepsilon_v \frac{\partial \bar{U}_{0v_0}}{\partial \bar{\xi}} = O\left(\theta\, |\bar{U}^{**}_{0v_{02}} - \bar{U}^{**}_{0v_{01}}|^2\right), \tag{204}$$

so that eq.(172) is fully verified as a consequence of $\theta \ll 1$. A final question is the following: Is it truly legitimate to neglect the dissipation, since this is used only for defining the internal structure of shocks which could be /V,82

schematized by compression discontinuity surfaces, even in the case in which $\bar{\eta} \to \infty$? Let us define the problem: Equation (202) demonstrates that the thickness of the shocks varies in inverse proportion to the discontinuity of U; however, it is intuitively clear (which we will demonstrate below) that this discontinuity, for a given shock, weakens progressively as time elapses, i.e., as $\bar{\eta}$ increases; in fact, the discontinuity even tends to zero as $\bar{\eta} \to \infty$. This means that the thickness of a given shock increases with $\bar{\eta}$ and even increases beyond all limits as soon as $\bar{\eta} \to \infty$. Thus, it could well be that the internal structures of the shocks finally invade the entire proximal domain. So as to make certain that this is not the case, we will return to eq.(186) and study its behavior when $\bar{\eta} \to \infty$, still assuming that $\frac{\epsilon_v}{\theta}$ is small. Let us return to the study of F; let F^* be the absolute minimum, so that

$$F - F^* = \int_{\bar{\xi}^*}^{\bar{\xi}_1} \bar{U}_{0v_0}(\bar{\xi}_2, \bar{\eta}_0)\, d\bar{\xi}_2 + \frac{2}{\Gamma_0} \frac{\bar{\xi}_1 + \bar{\xi}^* - 2\bar{\xi}}{\bar{\eta} - \bar{\eta}_0} (\bar{\xi}_1 - \bar{\xi}^*), \tag{205}$$

and let us define

$$I = \operatorname*{Max}_{\bar{\xi}_1} \left\{ \left| \int_{-\infty}^{\bar{\xi}_1} \bar{U}_{0v_0}(\bar{\xi}_2, \bar{\eta}_0)\, d\bar{\xi}_2 \right| \right\}. \tag{206}$$

It is then obvious that

$$F = I\varphi + \frac{2}{\Gamma_0} \frac{(\bar{\xi}_1 - \bar{\xi})^2}{\bar{\eta} - \bar{\eta}_0}, \tag{207}$$

where φ is comprised between -1 and +1. Equation (207) shows that /V,83

$$\frac{(\bar{\xi}^* - \bar{\xi})^2}{\bar{\eta} - \bar{\eta}_0} \leq \Gamma_0 I, \tag{208}$$

so that we can write

$$\frac{(\bar{\xi}^* - \bar{\xi})^2}{\bar{\eta} - \bar{\eta}_0} = \Gamma_0 I \psi \qquad |\psi| \leq 1 \tag{209}$$

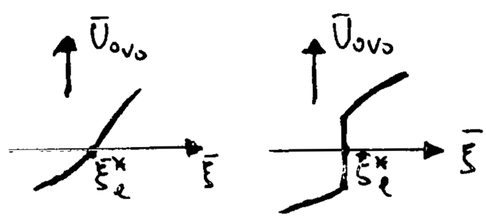
and that eq.(205) will assume the following form:

$$F-F^* = \int_{\bar{\xi}^*}^{\bar{\xi}_1} \bar{U}_{o v_o}(\bar{\xi}_2, \bar{\eta}_o)\, d\bar{\xi}_2 + 4\sqrt{\frac{I\gamma}{\Gamma_o(\bar{\eta}-\bar{\eta}_o)}}\,(\bar{\xi}_1 - \bar{\xi}^*) + \frac{2}{\Gamma_o}\frac{(\bar{\xi}_1 - \bar{\xi}^*)^2}{\bar{\eta}-\bar{\eta}_o}. \tag{210}$$

At the limit, we have

$$0 \leqslant F - F^* = \int_{\bar{\xi}_l^*}^{\bar{\xi}_1} \bar{U}_{o v_o}(\bar{\xi}_2, \bar{\eta}_o), \tag{211}$$

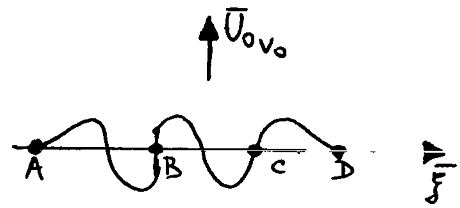
denoting by $\bar{\xi}_l^*$ the limit of $\bar{\xi}^*$, if it does exist. The accompanying diagram shows the two cases that might occur if $\bar{U}_{o_{v_o}}$ is stepwise continuous: The



quantity $\bar{\xi}_l^*$ is located in one of the intersections of the graph $(\bar{U}_{o_{v_o}}, \bar{\xi})$ with $\bar{U}_{o_{v_o}} = 0$, where the direction of passage is $\bar{U}_{o_{v_o}} < 0$ toward $\bar{U}_{o_{v_o}} > 0$ on increasing $\bar{\xi}$. Equation (211) shows that $\bar{\xi}_l^*$ ensures a relative maximum of

$$J = -\int_{-\infty}^{\bar{\xi}_1} \bar{U}_{o v_o}(\bar{\xi}_2, \bar{\eta}_o)\, d\bar{\xi}_2, \tag{212}$$

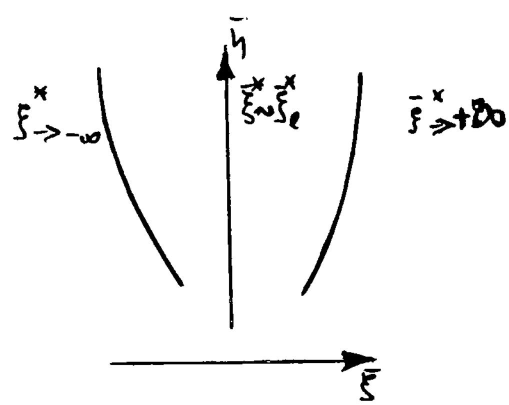
which must even be an absolute maximum if it is remembered that eq.(210) /V,84 is valid without having $\bar{\xi}_1 - \bar{\xi}^*$ close to zero. This condition of the maximum permits selecting the most convenient intersection, at least so far as the intersections inside the graph are concerned. In the case of the accompanying diagram, four intersections, A and D at the extremities and B and C at the interior, exist; the intersection to be selected depends on the areas of the successive loops. However, it should be mentioned that, in the general case, the passage to the limit $\bar{\xi} \to \infty$, $\bar{\eta} \to \infty$ is nonuniform: For any finite $\bar{\xi}$, we find

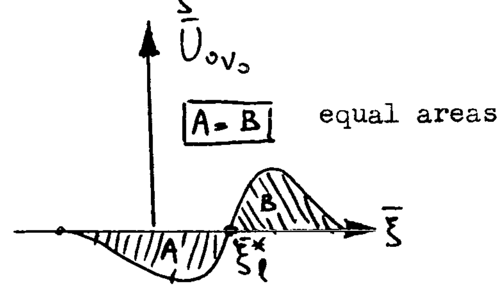


that $\xi^*(\bar{\xi}, \bar{\eta}) \to \xi_\ell^*$ if $\bar{\eta} \to \infty$; however, it is not true that $\xi^* \to \xi_\ell^*$ if simultaneously $\bar{\xi} \to \infty$ and $\bar{\eta} \to \infty$ since eq.(208) shows that ξ^* certainly does not tend toward ξ_ℓ^* if $\bar{\xi}$ and $\bar{\eta} \to \infty$, in such a manner that

$$\frac{(\xi_\ell^* - \bar{\xi})^2}{\bar{\xi} - \bar{\xi}_0} \geqslant \Gamma_0 \underline{T}; \tag{213}$$

whereas this is so if $\xi^* \to \pm\infty$. The three limiting values of ξ^* are thus in fact, $-\infty$, ξ_ℓ^*, and $+\infty$, assuming that a single value ξ_ℓ^* exists which ensures the





absolute maximum of J.

Let us treat here the case in which the graph of $\bar{U}_{0v_0}$ has the slope shown

in the accompanying diagram. The condition (189a), defining $\bar{\xi}^*(\bar{\xi}, \bar{\eta})$ can be given the following form, provided that $\bar{\xi}^*$ is close to $\bar{\xi}_\ell^*$:

/V,85

$$(\bar{\xi}^* - \bar{\xi}_\ell^*)\, U'_{0v_0}(\bar{\xi}_\ell^*) \cong \frac{4}{\Gamma_0} \frac{(\bar{\xi} - \bar{\xi}_\ell^*)}{\bar{\eta} - \bar{\eta}_0}, \tag{214}$$

and the corresponding value of F^* will then be

$$F^* \cong -I + \frac{2}{\Gamma_0} \frac{(\bar{\xi} - \bar{\xi}_\ell^*)^2}{\bar{\eta} - \bar{\eta}_0}, \tag{215}$$

which represents the absolute minimum of F only if

$$(\bar{\xi} - \bar{\xi}_\ell^*)^2 \leq \frac{\Gamma_0}{2} I (\bar{\eta} - \bar{\eta}_0). \tag{216}$$

Conversely, if we have

$$(\bar{\xi} - \bar{\xi}_\ell^*)^2 \geqslant \frac{\Gamma_0}{2} I (\bar{\eta} - \bar{\eta}_0), \tag{217}$$

we must set

$$\xi^* = \bar{\xi}, \qquad F^* = 0. \tag{218}$$

If the initial profile $\bar{U}_{0v_0}(\bar{\xi})$ satisfies the condition $\int_{-\infty}^{\infty} U_{0v_0}(\bar{\xi})d\bar{\xi} = 0$ and if the slope shown in the accompanying diagram exists, then the asymptotic behavior of the function $U_{0v}(\bar{\xi}, \bar{\eta})$ will be given by

$$\bar{U}_{0v_0}(\bar{\xi}, \bar{\eta}) = \sqrt{\frac{8I}{\Gamma_0(\bar{\eta} - \bar{\eta}_0)}}\; \frac{\bar{\xi} - \bar{\xi}_\ell^*}{\sqrt{\frac{\Gamma_0}{2} I (\bar{\eta} - \bar{\eta}_0)}} \Big\{ \mathcal{H}_v\Big(\bar{\xi} - \bar{\xi}_\ell^* + \sqrt{\tfrac{\Gamma_0}{2} I (\bar{\eta} - \bar{\eta}_0)}\Big) - \mathcal{H}_v\Big(\bar{\xi} - \bar{\xi}_\ell^* - \tfrac{\Gamma_0}{2} I (\bar{\eta} - \bar{\eta}_0)\Big)\Big\}, \tag{219}$$

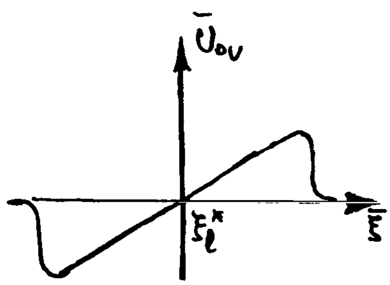
yielding

$$\mathcal{H}_v(z) = \frac{1}{1 + \dfrac{1}{\sqrt{1+\frac{\Gamma_0}{4}\bar{U}'_{0v_0}(\xi_\ell^*)(\bar{\zeta}-\bar{\zeta}_0)}}\exp\left\{\dfrac{\theta}{\varepsilon_v}\left(1+\dfrac{\kappa_0}{\chi_0}\right)\sqrt{\dfrac{2I\Gamma_0}{\bar{\zeta}-\bar{\zeta}_0}}\; z\right\}} \tag{220}$$

When $\varepsilon_v = 0$, the function $\mathfrak{H}_v(z)$ will reduce to the Heaviside function /V,86

$$\mathcal{H}(z) = \begin{cases} 0 & \text{si } z < 0 \\ 1 & \text{si } z > 0 \end{cases} \tag{221}$$

The profile of $U_{0_{v_0}}$, as soon as $\bar{\eta} \to \infty$, tends toward a profile in N^-, where the



width of N^- is

$$2\sqrt{\frac{\Gamma_0}{2} I (\bar{\zeta}-\bar{\zeta}_0)}\ , \tag{222}$$

while the amplitude reads

$$\sqrt{\frac{8\,I}{\Gamma_0(\bar{\zeta}-\bar{\zeta}_0)}} \tag{223}$$

However, the points of N^- are rounded off by the internal structure of the shocks whose thickness is

$$O\left(\frac{\varepsilon_v}{\theta}\frac{\chi_0}{\kappa_0+\chi_0}\sqrt{\frac{\bar{\zeta}-\bar{\zeta}_0}{2I\Gamma_0}}\right). \tag{224}$$

The thickness in question thus increases with $\bar{\eta}$ but, relative to the width of N^-,

it does not increase and remains $O\left(\frac{\epsilon_v}{\theta} \frac{x_0}{K_0 - x_0} \frac{1}{2IP_0}\right)$.

5.3 Sonic Boom Produced by Supersonic Flight

Making use of eqs.(1.23), changing eq.(1.25), and continuing the analysis given in Section 5.1.2 while taking the nonhomogeneity terms into consideration, it becomes obvious that these must be counted among the terms in ℓ/L if ℓ is smaller than the nonhomogeneity scale of the atmosphere. Here, this term denotes a length H such that, for example, $\frac{H|\nabla\rho_0|}{\rho_0} = O(1)$. In the applica- /V,87 tions we wish to make here, H and L are of the same order of magnitude and $\ell \ll \{L, H\}$. The formal statement of the theorem 2 remains valid under the condition that, in the linear terms of eqs.(178) and (182), the nonhomogeneity terms are included. These can be readily obtained from eqs.(31) of Section 4.2 by disregarding the derivatives in x_1 of quantities of rank 1, i.e., the boxed terms, and by replacing the tensor H by the unit tensor. A detailed calculation will be given in an ONERA paper to be published in 1964. Here, we are exclusively interested in the fate of eq.(178). For this, it is sufficient to replace, in the equation in question, the term $\frac{\partial v_1}{\partial x_0} + \frac{1}{2} V_1 K$ by the first term of eq.(140) of Section 4.2.

Proposition 1: The notations are those given in theorem 2 of the present Chapter and those of theorem 12 of Chapter IV with respect to $\mathfrak{A}$ and $\mathfrak{p}$, where the width of the wave train is ℓ, the length covered along the rays is L, and the nonhomogeneity scale is $\mathfrak{H}$ [defined by the condition $\frac{H|\mathbf{V}\rho_0|}{\rho_0}, \dots \frac{H}{|V_0|} \left|\frac{\partial \mathbf{V}_0}{a_0 \partial t}\right| = O(1)$]; we then assume

$$\ell \ll L, \quad \ell \ll H. \tag{1}$$

Under these conditions and with a relative error of $O\left(\frac{\ell}{L} + \frac{\ell}{H}\right)$, /V,88

$$\begin{cases} \mathbb{V} = \mathbb{V}_0 + V_1(x_0, x_1; \mathcal{R})\, \mathfrak{n}(x_0; \mathcal{R}), \\ p = p_0 + \rho_0 c_0 V_1, \\ \rho = \rho_0 + \rho_0 \dfrac{V_1}{c_0}, \\ S = S_0, \end{cases} \tag{2}$$

is an approximate solution of the equations of motion, provided that

$$\rho_0 \, v_1 \frac{\partial v_1}{\partial x_1} + \sqrt{\frac{|\mathbf{p}|}{\rho_0 A}} \frac{\partial}{\partial x_0}\left\{\sqrt{\frac{\rho_0 A}{|\mathbf{p}|}} \, v_1\right\} + \\ + \frac{v_1}{2}\left\{\mathbf{n}\cdot(\nabla \mathbf{V}_0)\cdot\mathbf{n} + (\Gamma_0 - 1)\, \nabla\cdot\mathbf{V}_0\right\} = 0 . \tag{3}$$

Let us now investigate how to make allowance for the dissipation phenomena; for this, let us start from the equations of motion under the form

$$\left\{\begin{array}{l} \mathcal{C} \equiv \frac{\partial \rho}{\partial t} + \mathbf{V}\cdot\nabla\rho + \rho\,\nabla\cdot\mathbf{V} = 0 \\ \mathbb{M} \equiv \rho \frac{\partial \mathbf{V}}{\partial t} + \rho\mathbf{V}\cdot\nabla\mathbf{V} + \nabla p + \rho\mathbf{g} + \nabla\cdot\boldsymbol{\tau} = 0 \\ \Sigma \equiv \rho T \frac{\partial s}{\partial t} + \rho T\,\mathbf{V}\cdot\nabla s + T\,\nabla\cdot\left(\frac{\mathbf{Q}}{T}\right) - \rho T \mathcal{D} = 0 \end{array}\right. \tag{4}$$

with

$$\left\{\begin{array}{l} \mathbf{Q} = -k\,\nabla T \quad , \quad \boldsymbol{\tau} = -2\mu\,\mathbb{D} - (\mu_v - \tfrac{2}{3}\mu)\,\mathbf{I}\,(\nabla\cdot\mathbf{V}) \\ \rho\mathcal{D} = 2\mu\,\frac{\mathbb{D}:\mathbb{D}}{T} + \frac{\mu_v - \frac{2}{3}\mu}{T}\,|\nabla\cdot\mathbf{V}|^2 + \frac{k|\nabla T|^2}{T^2} \end{array}\right. \tag{5}$$

and

$$\mathbb{D} = \frac{1}{2}\left\{\nabla\mathbf{V} + (\nabla\mathbf{V})^{Tr}\right\} . \tag{6}$$

In Section 5.1 we made use of the same equations (4) but with $\boldsymbol{\tau} = \mathbf{Q} = \mathcal{D} = 0$ and formed the following combination: /V,89

$$\frac{1}{2\rho_0}\left(c_0\,\mathcal{C} + \mathbf{n}\cdot\mathbb{M} + \frac{g_{s_0}}{\rho_0 c_0 T_0}\,\Sigma_1'\right) = 0 \tag{7}$$

It is exactly this combination that leads to eq.(3) by the indicated mechanism if the dissipative phenomena are neglected. Let us investigate the modification produced by the presence of dissipative terms. Since μ, μ_v, and k are small, it is obvious that, in the explicit writing of these terms, we must retain only the derivatives in x_1 of the magnitude of rank 1, i.e.,

$$\begin{cases} \mathbb{T} = -\left\{2\mu \dfrac{\partial v_1}{\partial x_1} + \left(\mu_v - \dfrac{2}{3}\mu\right)\dfrac{\partial v_1}{\partial x_1}\right\}\mathbb{m}\,\mathbb{m} + O\left(\dfrac{\mu V_1}{L}\right) + O\left(\dfrac{\mu V_1}{H}\right). \\ \mathbb{Q} = -k\,\mathbb{m}\,\dfrac{\partial T_1}{\partial x_1} + O\left(\dfrac{k T_1}{L}\right) + O\left(\dfrac{k T_1}{H}\right), \end{cases} \tag{8}$$

Then,

$$\begin{cases} (\nabla\cdot\tau)\cdot\mathbb{m} \cong -\left(\mu_v + \dfrac{4}{3}\mu_0\right)\dfrac{\partial^2 v_1}{\partial x_1^2} \\ T\,\nabla\cdot\dfrac{\mathbb{Q}}{T} \cong -k_0\,\dfrac{\partial^2 T_1}{\partial x_1^2} \end{cases} \tag{9}$$

and, finally,

$$\rho T \mathcal{D} \cong 2\mu_0\left(\frac{\partial v_1}{\partial x_1}\right)^2 + \left(\mu_{v_0} - \frac{2}{3}\mu_0\right)\left(\frac{\partial v_1}{\partial x_1}\right)^2 + \frac{k_0}{T_0}\left(\frac{\partial T_1}{\partial x_1}\right)^2, \tag{10}$$

so that the combination corresponding to eq.(7) will yield

$$\begin{aligned} \mathbb{m}_0\,(\nabla\cdot\tau) + \frac{g_{s_0}}{\rho_0 c_0 T_0}\left\{T\,\nabla\cdot\left(\frac{\mathbb{Q}}{T}\right) - \rho T \mathcal{D}\right\} \\ = -\left(\mu_{v_0} + \frac{4}{3}\mu_0\right)\frac{\partial^2 v_1}{\partial x_1^2} - \frac{k_0 g_{s_0}}{\rho_0 c_0 T_0}\frac{\partial^2 T_1}{\partial x_1^2} \\ - \frac{g_{s_0}}{\rho_0 T_0 c_0}\left(\mu_{v_0} + 4/3\,\mu_0\right)\left(\frac{\partial v_1}{\partial x_1}\right)^2 - \frac{k_0 g_{s_0}}{\rho_0 c_0 T_0^2}\left(\frac{\partial T_1}{\partial x_1}\right)^2. \end{aligned} \tag{11}$$

However, we know from Section 5.2.3 that the quadratic terms are negligible and that we can use the following relation: /V,90

$$T_1 \cong \frac{g_{s_0}}{\rho_0^2}\rho_1 \cong \frac{g_{s_0}}{\rho_0 c_0} v_1, \tag{12}$$

which results in

$$\mathbb{m}\cdot(\nabla\cdot\tau) + \frac{g_{s_0}}{\rho_0 c_0 T_0}\left\{T\,\nabla\cdot\frac{\mathbb{Q}}{T} - \rho T\mathcal{D}\right\} \cong \tag{13}$$

$$\cong \left(\mu_{v_0} + \frac{4}{3}\mu_0 + \frac{k_0 (g_{s_0})^2}{\rho_0^2 c_0^2 T_0}\right) \frac{\partial^2 v_1}{\partial x_1^2} .$$

For an ideal gas, we have

$$g_{s_0} = (\gamma - 1)\, \rho_0\, T_0 \tag{14}$$

and, consequently,

$$\mu_{v_0} + 4/3\, \mu_0 + \frac{k_0 (g_{s_0})^2}{\rho_0 c_0^2 T_0} = \mu_{v_0} + \frac{4}{3}\mu_0 + \frac{(\gamma-1) k_0}{c_p} . \tag{15}$$

Proposition 2: Taking the dissipative phenomena into consideration, eq.(3) of proposition 1 must be modified as follows:

$$\begin{aligned} \rho_0 v_1 \frac{\partial v_1}{\partial x_1} + \sqrt{\frac{|\mathfrak{p}|}{\rho_0 \mathcal{R}}} \frac{\partial}{\partial x_0}\left(\sqrt{\frac{\rho_0 \mathcal{R}}{|\mathfrak{p}|}}\, v_1\right) + \\ + \frac{v_1}{2}\left\{ \mathbf{n}\cdot(\bar{\nabla} \mathbf{V}_0)\cdot \mathbf{n} + (\Gamma_0 - 1)\, \bar{\nabla}\cdot \mathbf{V}_0 \right\} = \\ = \frac{\mu_{v_0} + \frac{4}{3}\mu_0 + \frac{k_0 (g_{s_0})^2}{\rho_0 c_0^2 T_0}}{2\rho_0} \frac{\partial^2 v_1}{\partial x_1^2} . \end{aligned} \tag{16}$$

It is logical to proceed here as we had done in the case of plane waves, i.e., to complement eq.(16) by the other equations of motion and to verify, a posteriori, eqs.(12) and (13); we will be satisfied here with the data obtained in the case of plane waves. /V,91

Before continuing, we will reduce eq.(16) to the canonical Burgers form, by immediately picking up the specific case in which $x_1 = 0$ is the Mach wave associated with supersonic flight. We will use the variables x_1, T; $\mathfrak{R}$ of Section 4.3.2, and will pose

$$v_1 = V_a\, \mathcal{M}_a^{3/2} (\mathcal{M}_a^2 - 1)(2\, c_{0_a}\, T)^{-\frac{1}{2}} \left(\mathcal{B}(T; \mathfrak{R})\right)^{-1} W \tag{17}$$

Then, eq.(16) becomes

$$W \frac{\partial w}{\partial x_1} + \frac{\partial w}{\partial y_1} = \nu_e \frac{\partial^2 w}{\partial x_1^2} \tag{18}$$

under the condition of posing

$$\left\{\begin{aligned} y_1 &= V_a \mathcal{M}_a^{3/2}(\mathcal{M}_a^2-1)\int_0^{T} \Gamma_0 (2c_{0a} T_1)^{-\frac{1}{2}} \mathcal{B}^{-1}(T_1, \mathcal{R})\, dT_1, \\ \nu_e &= \frac{\mu\nu_0 + \frac{4}{3}\mu_0 + \frac{k_0 (g_{s_0})^2}{\rho_0 c_0^2 T_0}}{2\rho_0 V_a \Gamma_0 \mathcal{M}_a^{3/2}(\mathcal{M}_a^2-1)(2c_{0a} T)^{-1/2}\mathcal{B}^{-1}(T;\mathcal{R})} . \end{aligned}\right. \tag{19}$$

We suggest to the reader to determine, for exercise, the dimensions of W, x_1, y_1, ν_e. From the investigation made on plane waves, we know that the function W must be identical to the function $F(-\mu_a x_1)$ of proposition 16 in Chapter IV as soon as $\frac{c_0 T}{\ell_a} = O(\epsilon^{-\frac{1}{2}})$, denoting by ℓ_a the length of the aircraft and by ϵ the order of magnitude of the perturbations (in relative value) in the vicinity of the latter. However, the proposition 16 of Chapter IV must be doubted as /V,92 soon as $\frac{c_0 T}{\ell_a} = O(\epsilon^{-1})$, which means that the connectivity zone is located in $\frac{c_0 T}{L} = O(\epsilon^{+\frac{1}{2}})$ and that, consequently, we must identify W with $F(-\mu_a x_1)$ in $y_1 =$ $= 0$.

Proposition 2: The notations used are those of proposition 16 in Chapter IV, posing

$$y_1 = \frac{\gamma+1}{2} V_a \mathcal{M}_a^{3/2}(\mathcal{M}_a^2-1)\int_0^{T} \frac{dT_1}{(2c_{0a} T_1)^{\frac{1}{2}} \mathcal{B}(T_1;\mathcal{R})},$$

$$\nu_e = \frac{\mu\nu_0 + \frac{4}{3}\mu_0 + (\gamma-1)k_0/c_p}{(\gamma+1)\rho_0 V_a \mathcal{M}_a^{3/2}(\mathcal{M}_a^2-1)(2c_{0a} T)^{-1/2}\mathcal{B}^{-1}(T;\mathcal{R})} \tag{20}$$

We note that $W(T, x_1; \mathcal{R})$ is the solution of the equation

$$W\frac{\partial W}{\partial x_1} + \frac{\partial W}{\partial y_1} = \nu_e \frac{\partial^2 W}{\partial x_1^2}, \tag{21}$$

which satisfies the initial condition

$$W(0, x_1; \mathcal{R}) = F(-\mathcal{M}_a x_1) \tag{22}$$

Under these conditions, the proximal field of perturbations produced by the aircraft is given by

$$\left\{\begin{array}{l} \mathbb{V} = \mathbb{V}_0 + c_0 V_1 \mathfrak{n} \\ p = p_0 + \rho_0 c_0 V_1 \\ \rho = \rho_0 + \dfrac{\rho_0}{c_0} V_1 \\ V_1 = V_a(R)\, \mathcal{M}_a^{3/2}(R) \left(\mathcal{M}_a^2(R) - 1\right) \left(2 c_a(R)\, T\right)^{-\frac{1}{2}} \mathcal{B}^{-1}(T; R)\, W(T, x_q; R). \end{array}\right. \tag{23}$$

Here, we will not return to an integration of Burgers' equation and rather will directly calculate the asymptotic behavior. Equation (7) of Section 4.3 is written as /V,93

$$2\pi F(\xi) = \frac{\partial^2}{\partial \xi^2} \int_{-\infty}^{\xi} \frac{S_e(\xi_1)}{\sqrt{\xi - \xi_1}}\, d\xi_1 \tag{24}$$

from which it follows that

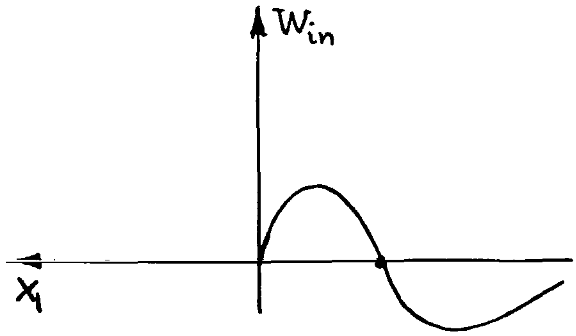
$$\begin{aligned} 2\pi \int_{-\infty}^{\infty} F(\xi)\, d\xi &= \lim_{\xi \to +\infty} \frac{\partial}{\partial \xi} \int_{-\infty}^{\xi} \frac{S_e(\xi_1)}{\sqrt{\xi - \xi_1}}\, d\xi_1 \\ &= \lim_{\xi \to +\infty} \int_{-\infty}^{\xi} \frac{S_e'(\xi_1)}{\sqrt{\xi - \xi_1}}\, d\xi_1 . \end{aligned} \tag{25}$$

If the integral of $S_e'(\xi_1)$ converges, i.e., if $S_e(\xi)$ tends toward a limit as soon as $\xi \to +\infty$ or even if S_e increases less rapidly than $\sqrt{\xi}$, we have

$$\int_{-\infty}^{\infty} F(\xi)\, d\xi = 0 . \tag{26}$$

We postulate the hypothesis that this is so, without disregarding the fact that the existence of the wake formed downstream of the aircraft raises a certain difficulty. In addition, it is almost certain that the wake cannot be uniquely represented by an equivalent fuselage, at least so far as the production of a sound field in the proximity of the aircraft is concerned; it is not known whether this is the same for the far field. Let us state that the measurements of the proximal sound field, in the sense intended here, confirm that the wake has a significance which possibly is nonnegligible but certainly is secondary.

Therefore, we adopt a hypothesis according to which eq.(26) actually takes place. The accompanying diagram shows a typical configuration of the /V,94



graph of $W_{1n} = W(0, x_1)$ for an aircraft. We know that the asymptotic behavior of $W(y_1, x_1)$ for $y_1 \to \infty$ is given by the formula

$$W(y_1, x_1) = \sqrt{\frac{2I}{y_1}}\,\frac{x_1 - x_{1e}^*}{\sqrt{2I\,y_1}}\left\{\mathcal{H}\left(x_1 - x_{1e}^* + \sqrt{2I\,y_1}\right) - \mathcal{H}\left(x_1 - x_{1e}^* - \sqrt{2I\,y_1}\right)\right\}, \tag{27}$$

denoting by $\mathfrak{H}$ the Heaviside function if ν_e is neglected. If, for taking ν_e into consideration, eq.(2.220) is applied, we obtain the same formula as eq.(27) with $\mathfrak{H}_v(z)$ instead of $\mathfrak{H}(z)$ and

$$\mathcal{H}_v(z) = \frac{1}{1 + \exp\left\{-\sqrt{\frac{2I}{y_1}}\,\frac{z - z_c}{2\nu_e}\right\}}. \tag{28}$$

However, the fact that ν_e is variable makes it impossible to obtain an accurate definition of z_c. Nevertheless, a specialized investigation (see the announced ONERA publication) makes it possible to demonstrate that

$$\pm \nu_e z_c \sqrt{\frac{2I}{y_1}} + \int_0^{y_1} \frac{\nu_e(t)\,W'_{in}(x_{1e}^*)}{1 + t\,W'_{in}(x_{1e}^*)}\,dt = 0, \tag{29}$$

where the ± signs must be associated, respectively, with the upstream (+) and downstream (-) points of N. /V,95

Below, we will neglect the internal structure of the shocks since we can always superpose these afterwards because of the fact that the thickness is negligible relative to the length of N. This latter will be characterized by its amplitude $\sqrt{\frac{2I}{y_1}}$ and by its width $\sqrt{2Iy_1}$. Let us recall that I is given by

$$I = \underset{X_1}{\mathrm{Max}} \int_{X_1}^{\infty} W_{in}(z)\, dz = \underset{\xi}{\mathrm{Max}} \left\{ \mathcal{M}_a^{-1} \int_{-\infty}^{\xi} d\xi_1 \frac{\partial}{\partial \xi_1} \int_{-\infty}^{\xi_1} \frac{S_e'(\xi_2)}{\sqrt{\xi_1 - \xi_2}} d\xi_2 \right\}$$
$$= \underset{\xi}{\mathrm{Max}} \left\{ \frac{1}{\mathcal{M}_a} \int_{-\infty}^{\xi} \frac{S_e'(\xi_1)}{\sqrt{\xi - \xi_1}} d\xi_1 \right\}. \tag{30}$$

Dimensionally, we have I = (length)$^{3/2}$ so that, denoting by $\mathcal{V}_a$ the total volume of the aircraft and by ℓ_a its length, we can pose

$$K_F(R)\, \mathcal{V}_a\, \ell_a^{-3/2} = \underset{\xi}{\mathrm{Max}} \int_{-\infty}^{\xi} \frac{S_e'(\xi_1, R)}{\sqrt{\xi - \xi_1}} d\xi_1 \tag{31}$$

where K_F is a form factor which depends on the geometry of the aircraft and on the distribution of the loads. Knowing these elements, it is possible to calculate S_e by means of eq.(74) of Chapter I. It may be of interest to separate the effects of volume and lift, by writing

$$S_a = S_e^{(flight)} + S_e^{(port)}, \tag{32}$$

so that eq.(31) can be written for only the volume effects, and then to write

$$K_F^{(port)}(R)\, \frac{P_a}{\rho_0 V_a^2}\, \ell_a^{\frac{1}{2}}\, d = \underset{\xi}{\mathrm{Max}} \int_{-\infty}^{\xi} \frac{S_e'^{(port)}(\xi_1, R)}{\sqrt{\xi - \xi_1}} d\xi_1 \tag{33}$$

such that, for the overall unit, /V,96

$$2\pi \mathcal{M}_a I = K_F^{(flight)}(R)\, \mathcal{V}_a\, \ell_a^{-3/2} + K_F^{(port)}(R)\, \frac{P_a}{\rho_a V_a^2}\, \ell_a^{-\frac{1}{2}} = \underset{\xi}{\mathrm{Max}} \int_{-\infty}^{\xi} \frac{S_e'(\xi, R)}{\sqrt{\xi - \xi_1}} d\xi_1, \tag{34}$$

where $\ell_a d$ denotes the area of the planform of the wing and P_a the weight of the aircraft.

Let us now attempt to express the amplitude and the width of N as a function of the flight characteristics (aircraft geometry, law of lift, kinematics of flight) and of the distance covered along the sound ray. Let us start with the case of a rectilinear and uniform flight in a homogeneous atmosphere. In this case, $\mathfrak{B} \equiv 1$ and eq.(19) will yield

$$y_1 = V_a \mathcal{M}_a^{3/2} (\mathcal{M}_a^2 - 1)(\gamma+1)\, 2^{-\frac{1}{2}}\, c_{0_a}^{-\frac{1}{2}}\, T^{\frac{1}{2}} \tag{35}$$

in such a manner that, denoting by V_1^* the amplitude of N and by Λ^* its width, we have

$$\left\{ \begin{aligned} \frac{V_1^*}{c_0} &= (\gamma+1)^{-1/2} \pi^{-1/2} 2^{-1/4} \mathcal{M}_a^{3/4} (\mathcal{M}_a^2 - 1)^{-1/4} \cdot \\ &\quad \left(K_F^{(Vol)} \mathfrak{V}_a + K_F^{(Port)} \frac{P_a \ell_a}{\rho_a V_a^2}\right)^{\frac{1}{2}} (\ell_a c_0 T)^{-3/4} \\ \Lambda^* &= (\gamma+1) \frac{V_1^*}{c_0}\, c_0 T . \end{aligned} \right. \tag{36}$$

In order to pass to the case of a nonhomogeneous atmosphere and a random flight, no calculation is necessary to obtain the following proposition.

<u>Proposition 3</u>: The notations of proposition 16 of Chapter IV are retained here. For each ray $\mathfrak{R}$, with the equivalent fuselage denoted by $S_e(\xi, \mathfrak{R})$, we define the form factors $K_F^{(flight)}(\mathfrak{R})$ and $K_F^{(port)}(\mathfrak{R})$, depending on the ray, by

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$$K_F^{(Vol)}(\mathfrak{R})\, \mathfrak{V}_a\, \ell_a^{-3/2} + K_F^{(Port)} \frac{P_a}{\rho_a V_a^2}\, \ell_a^{-1/2} = \underset{\xi}{\mathrm{Max}} \int_{-\infty}^{\xi} \frac{S_e'(\xi_1, \mathfrak{R})}{\sqrt{\xi - \xi_1}}\, d\xi_1 , \tag{37}$$

denoting by $\mathfrak{V}_a$ the volume of the aircraft, by ℓ_a its length, and by P_a its weight. We assume as known the function $\mathfrak{B}(T, \mathfrak{R})$, connected to the geometry of the rays $\mathfrak{R}$ and to the law of density variation in the atmosphere. Under these conditions, the proximal perturbation field is characterized by a value N

$$V_1 = c_0 \mathcal{J} \frac{x_1}{\Lambda} \left\{ H(x_1 + \Lambda) - H(x_1 - \Lambda) \right\} , \tag{38}$$

denoting by H(z) the Heaviside function. Here, $\mathfrak{J}$ is the amplitude of nondimensionality of N in the sense that $\mathfrak{J}$ is equal to the maximum of $\frac{\Delta p}{\rho_0 c_0^2}$ at a given observation point during passage of the N, while ρ_0 and c_0 represent the specific mass and the velocity of sound in a nonperturbed atmosphere under the observation conditions; Λ is the half-width of N. This yields

$$\mathfrak{J} = (\gamma+1)^{-1/2} n^{-1/2} 2^{-1/4} \mathcal{M}_a^{3/4} (\mathcal{M}_a^2 - 1)^{-1/4} \left\{ K_F^{(Vol)} \mathcal{D}_a + K_F^{(Port)} \frac{P_a \ell_a}{\rho_a V_a^2} \right\}^{\frac{1}{2}} \cdot \tag{39}$$
$$\left(\ell_a c_{o_a} T \right)^{-3/4} \frac{T^{\frac{1}{4}}}{\mathcal{B}(T;R)} \left(\int_0^T \frac{dT_1}{2T_1^{1/2}\mathcal{B}(T_1;R)} \right)^{-1/2}$$

$$\Lambda = (\gamma+1)\,\mathfrak{J}\, c_{o_a} T \quad \frac{\mathcal{B}(T;R)}{T^{1/2}} \int_0^T \frac{dT_1}{2\,T_1^{1/2}\mathcal{B}(T_1;R)} \,. \tag{40}$$

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